A limit theorem for occupation times of Lamperti's stochastic processes

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Abstract: The arcsine law for random walks on the line is well known and it was extended greatly by Lamperti for a class of discrete-time stochastic processes. In the present paper we treat its extreme case where the excursion intervals have very heavy tail probabilities. The result is a refinement of Lamperti's theorem. A functional limit theorem is also discussed.

Key words: occupation times; extremal process; arcsine law; slowly varying function.

1. Introduction. It is well known that the ratio of the time the simplest random walk spends on the positive side obeys the arcsine law in the long term. Lamperti [8] extended this result for the following class of discrete-time stochastic processes: Let $X = \{X(n)\}_{n\geq 0}$ be a stochastic process whose state space S is divided into S_+, S_- and a one-point set $\{\sigma\}$. Assume that the process can get from one of S_+ and S_- to the other only by passing through σ. It is not necessary that X has the Markov property, but instead we assume that the process visits the state σ infinitely many times with probability one and starts afresh whenever it visits σ.

Let $A(n)$ denote the occupation time of the set S_+ up to time n. The time spent on σ is counted or not according to whether the last state occupied was in S_+ or not although this is not essential in the present paper.

Lamperti [8] showed that the class of possible limiting random variables in law of $A(n)/n$ as $n \to \infty$ is $\{Y_{p,\alpha};\, 0 \leq p \leq 1,\, 0 \leq \alpha \leq 1\}$: $Y_{p,\alpha}$ is a $[0,1]$–valued random variable with the Stieltjes transform given by

$$E\left(\frac{1}{\lambda + Y_{p,\alpha}}\right) = \frac{p(\lambda+1)^{\alpha-1} + (1-p)\lambda^{\alpha-1}}{p(\lambda+1)^{\alpha} + (1-p)\lambda^{\alpha}},$$

for $\lambda > 0$. If $0 < p < 1, 0 < \alpha < 1$, then the distribution is continuous and the density is known explicitly. He also obtained a necessary and sufficient condition for the convergence in law to $Y_{p,\alpha}$. Especially, when $0 < p < 1$ and $0 \leq \alpha < 1$, the condition can be rewritten as follows: τ denoting the first hitting time of X to σ (i.e., $\tau = \inf\{n \geq 1 \,|\, X_n = \sigma\}$),

$$P_\sigma(\tau > n, X(1) \in S_+) \sim \frac{c_+}{n^\alpha L(n)}, \tag{1}$$

$$P_\sigma(\tau > n, X(1) \in S_-) \sim \frac{c_-}{n^\alpha L(n)} \tag{2}$$

as $n \to \infty$ for $c_+, c_- > 0$ and slowly varying $L(x)$. Here, '$\sim$' means that the ratio converges to 1. In this case it holds that $p = c_+/(c_+ + c_-)$. For details we refer to Y. Yano *et al.* [3], which discusses the functional limit theorem for Lamperti's theorem.

Although half a century has already passed since Lamperti's result, many authors have been interested in this kind of problems. Among them, S. Watanabe [12] studied the case of diffusions and obtained results which are quite similar to Lamperti's. See also Barlow *et al.* [1].

In the present paper we are interested in the extreme case $\alpha = 0$ and hence $Y_{p,\alpha}$ is a Bernoulli random variable: $P(Y_{p,0} = 1) = p, P(Y_{p,0} = 0) = 1-p$. In the case of diffusions this case is already discussed by [6], and a refinement of Lamperti's theorem was obtained under a suitable nonlinear normalization, which corresponds, roughly speaking, to limit theorems under the log–log scale. The aim of the present paper is to prove similar theorems for discrete-time processes of the Lamperti type instead of diffusions. Our main result is the following

Theorem 1. *Let $L(x)$ be a slowly varying function at ∞. If* (1) *and* (2) *hold with $\alpha = 0$ and $c_+, c_- > 0$, then*

$$\lim_{n\to\infty} P_\sigma\Big(L\big(A(n)\big)/L(n) \leq x\Big) = \begin{cases} \dfrac{c_- x}{c_- x + c_+}, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

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The proof will be given in Section 3. Here P_σ means the law conditioned that X starts at σ, although it is not essential at all.

2. Preliminaries. Let $\{Z(t)\}_{t\geq 0}$ be a symmetric Cauchy process with Lévy measure dx/x^2 and let us consider the Poisson point process defined by its jumps $\Delta Z_t := Z(t) - Z(t-0)$. Define nondecreasing processes $m_+ = \{m_+(t)\}_{t\geq 0}$ and $m_- = \{m_-(t)\}_{t\geq 0}$ by

$$m_+(t) = \sup_{0<s\leq t} \Delta Z_s, \quad t > 0,$$
$$m_-(t) = \sup_{0<s\leq t} (-\Delta Z_s), \quad t > 0$$

and $m_+(0) = m_-(0) = 0$. In other words m_+ is the maximum process of a Poisson point process with intensity dx/x^2. m_+ is often referred to as the (canonical) *extremal process*. m_- may be regarded as an independent copy of m_+. Note that m_+ and m_- are 1-self-similar; for every $c > 0$,

$$\{m_\pm(ct)\}_{t\geq 0} \stackrel{d}{=} \{c\, m_\pm(t)\}_{t\geq 0}.$$

It is easy to write down the finite-dimensional marginal distributions. Especially, we have

$$P(m_\pm(t) \leq a) = P(m_\pm^{-1}(a) \geq t) = e^{-t/a}$$

for $t \geq 0, a > 0$. Here and throughout f^{-1} denotes the right-continuous inverse of nondecreasing function f; i.e., $f^{-1}(t) = \inf\{x > 0 \mid f(x) > t\}$. Notice that

(3) $\quad f(a-0) \leq b \quad$ *if and only if* $\quad a \leq f^{-1}(b)$.

Now for given $c_\pm > 0$, let

(4) $\quad \xi_+(t) := c_+ m_+(t), \quad \xi_-(t) := c_- m_-(t)$

and define

$$\Xi(t) = \Xi(c_+, c_-; t) := \min\{t, \xi_+(\xi_-^{-1}(t))\}.$$

This process is in fact the same as the one which appeared in [6], where $\Xi(t)$ is represented in terms of Brownian motion. Although the proof of the equivalence is not difficult, we do not go into details since we shall not use it in the sequel.

Note that, since ξ_+ and ξ_- are 1-self-similar, so is Ξ; i.e.,

$$\{\Xi(ct)\}_{t\geq 0} \stackrel{d}{=} \{c\,\Xi(t)\}_{t\geq 0}, \quad c > 0.$$

The one-dimensional marginal distribution is given as follows:

Lemma 1. *For* $t \geq 0$,

$$P\big(\Xi(t) \leq x\big) = \begin{cases} \dfrac{c_- x}{c_- x + c_+ t}, & 0 \leq x < t, \\ 1, & x \geq t. \end{cases}$$

Proof. Since the assertion is trivial if $x \geq t$, let us consider the case where $0 \leq x < t$. Then $\Xi(t) \leq x$ holds if and only if $\xi_+(\xi_-^{-1}(t)) \leq x$. Since $\xi_+(.)$ does not have fixed discontinuities and furthermore ξ_+ is independent of ξ_-, it is easy to see that

$$\xi_+(\xi_-^{-1}(t)) = \xi_+(\xi_-^{-1}(t) - 0) \quad a.s.$$

Therefore,

$$P(\Xi(t) \leq x) = P\big(\xi_+(\xi_-^{-1}(t) - 0) \leq x\big).$$

This combined with (3) implies that

$$\begin{aligned} P\big(\Xi(t) \leq x\big) &= P\big(\xi_-^{-1}(t) \leq \xi_+^{-1}(x)\big) \\ &= P\big(m_-^{-1}(t/c_-) \leq m_+^{-1}(x/c_+)\big). \end{aligned}$$

Since $m_-^{-1}(t/c_-)$ and $m_+^{-1}(t/c_+)$ are independent and exponentially distributed with means t/c_- and t/c_+, respectively, we see that the extreme right-hand side equals

$$\frac{c_-/t}{(c_-/t) + (c_+/x)} = \frac{c_- x}{c_- x + c_+ t}.$$

□

Now let X be the process in Introduction and let τ_n denotes the time of n-th visit to the special state σ:

$$\tau_0 = 0, \ \tau_n = \inf\{k > \tau_{n-1} \mid X(k) = \sigma\} \ (n \geq 1).$$

Let

$$\begin{aligned} Y_+(n) &= (\tau_n - \tau_{n-1}) \cdot 1_{S_+}(X(\tau_n - 1)), \\ Y_-(n) &= (\tau_n - \tau_{n-1}) \cdot 1_{S_- \cup \{\sigma\}}(X(\tau_n - 1)) \end{aligned}$$

and put

$$T_+(t) = \sum_{1\leq k\leq t} Y_+(k), \ T_-(t) = \sum_{1\leq k\leq t} Y_-(k), \ (t \geq 0)$$

for $n = 0, 1, \ldots$. Thus $T_+(n)$ (or $T_-(n)$) denotes the time spent on S_+ (resp. on $S_- \cup \{\sigma\}$) during the first n excursions and it holds

$$T_+(t) + T_-(t) = \tau_{[t]}, \quad t \geq 0,$$

where $[t]$ denotes the integral part of t.

Let $D([0,\infty) : \mathbf{R})$ denote the càdlàg function space endowed with the Skorohod-Lindvall J_1-topology (see [2,9]).

Lemma 2. *As* $\lambda \to \infty$,

$$\left(\frac{1}{\lambda} L\big(T_+(\lambda t)\big), \frac{1}{\lambda} L\big(T_-(\lambda t)\big)\right) \xrightarrow{\mathcal{L}} \big(\xi_+(t), \xi_-(t)\big),$$

in $D([0,\infty):\mathbf{R})^2$*, where* ξ_+ *and* ξ_- *are the same as in* (4) *and* '$\xrightarrow{\mathcal{L}}$' *denotes the convergence in law.*

Proof. Notice that (1) and (2) (with $\alpha = 0$) may be rewritten as

$$P_\sigma(Y_\pm(1) > n) \sim \frac{c_\pm}{L(n)}, \quad n \to \infty.$$

The convergence in law of such i.i.d. random variables are already known (see [4,10]). See also [11]. Hence we know the convergence of each component and hence it remains to study the joint convergence. In other words, we need only to show the asymptotic independence between $\{T_+(\lambda t)\}_{t\ge 0}$ and $\{T_-(\lambda t)\}_{t\ge 0}$ as $\lambda \to \infty$: Notice that the asymptotic independence is *not* trivial because $\{(Y_+(n), Y_-(n))\}_n$ are i.i.d. random vectors but $Y_+(n)$ and $Y_-(n)$ are dependent. However, this difficulty can be removed, for example, by the following easy argument: Let $\{N(t)\}_{t\ge 0}$ be an usual Poisson process which is independent of $\{(Y_+(n), Y_-(n))\}_n$ (extending the probability space, if necessary) and has intensity 1 (i.e. $E[N(1)] = 1$). Then,

$$\left(\frac{1}{\lambda} L\big(T_+(\lambda t)\big), \frac{1}{\lambda} L(T_-(\lambda t))\right)$$

and

$$\left(\frac{1}{\lambda} L\big(T_+(N(\lambda t))\big), \frac{1}{\lambda} L\big(T_-(N(\lambda t))\big)\right) \tag{5}$$

have the same limiting processes. This assertion is clear from the law of large numbers, at least for all finite-dimensional marginal distributions. (In fact the result holds also in J_1-topology, although we need not this fact here). Therefore, it is sufficient to prove the asymptotic independence of the components in (5). However, they are in fact independent themselves. To see this fact notice that

$$\big\{\big(T_+(N(t)), T_-(N(t))\big)\big\}_{t\ge 0}$$

is a vector-valued compound Poisson process and the two components have no common discontinuities because $Y_+(n) > 0$ implies $Y_-(n) = 0$. This proves the independence. □

In the sequel we need that the inverse function $L^{-1}(x)$ is defined for all $x \ge 0$. Therefore, we put some additional conditions on the slowly varying function L. They are of course inessential. Since the left-hand side of (1) is nonincreasing in n, it is harmless to assume that L is nondecreasing when $\alpha = 0$. Furthermore, replacing L by another slowly varying function L^* such that $L(x) \sim L^*(x), x \to \infty$, we can, without loss of generality, further assume that $L(x)$ is defined on $[0,\infty)$ and is a strictly increasing, continuous function such that $L(0) = 0$. Hence, in what follows L^{-1} is the inverse function in the usual sense. A typical example for such L is $L(x) = \log(x+1)$ with $L^{-1}(x) = e^x - 1$.

Lemma 3. *As* $\lambda \to \infty$,

$$\frac{1}{\lambda} L\Big(T_-\big(T_+^{-1}\big(L^{-1}(\lambda t)\big)\big)\Big) \xrightarrow{f.d.} \xi_-(\xi_+^{-1}(t)),$$

where '$\xrightarrow{f.d.}$' *denotes the weak convergence of all finite-dimensional marginal distributions.*

Proof. We use the idea of [5]. Let Λ denote the totality of nondecreasing right-continuous functions $x(t), t \ge 0$ satisfying that $x(0) = 0$ and $x(\infty) = \infty$. If $x_n, y_n \in \Lambda (n \ge 1)$ converge respectively to $x, y \in \Lambda$ at all continuity points of x and y, respectively, then

$$y_n(x_n^{-1}(t)) \to y(x^{-1}(t)), \quad n \to \infty$$

for every $t > 0$ such that

$$y(x^{-1}(t-0)-0) = y(x^{-1}(t)).$$

Now notice that the inverse process of the first component in Lemma 2 is $(1/\lambda)T_+^{-1}\big(L^{-1}(\lambda t)\big)$. Therefore, substituting this process into the second component in Lemma 2, we can deduce that

$$\frac{1}{\lambda} L\Big(T_-\big(T_+^{-1}\big(L^{-1}(\lambda t)\big)\big)\Big) \xrightarrow{f.d.} \xi_-(\xi_+^{-1}(t)).$$

Of course we need to check the condition

$$\xi_-(\xi_+^{-1}(t-0)-0) = \xi_-(\xi_+^{-1}(t)), \quad a.s., \tag{6}$$

for every fixed $t > 0$. But this can easily be verified if we note that ξ_+^{-1} and ξ_- are mutually independent and either of them do not have any fixed discontinuities.

Here, we derived the convergence of finite-dimensional distributions of $Y_n(X_n^{-1}(t))$ from the weak convergence of $\{(X_n(t), Y_n(t))\}_{t\ge 0}$. For the topological basis of this kind of argument we refer to [7]. □

3. Functional limit theorem. In this section we give a functional limit theorem for Theorem 1. Suppose that the conditions in Theorem 1 are satisfied with a slowly varying function L.

Let $\{A(n)\}_{n\geq 0}$ be as in Section 1 and for real $t \geq 0$ we define $A(t)$ by the linear interpolation

$$A(t) = A(n) + (t-n)(A(n+1) - A(n)), \qquad n \leq t < n+1.$$

Another definition for $A(t)$ is also possible; $A(t) := A([t])$. However, since the difference between these two definitions is at most 1, there is no significant difference in our limit theorems.

Theorem 2. *Let L be a slowly varying function satisfying the supplementary conditions stated before Lemma* 3. *Then, under the assumptions in Theorem* 1,

$$\left\{\frac{1}{\lambda} L\big(\, A(L^{-1}(\lambda t))\,\big)\right\}_{t\geq 0} \xrightarrow{f.d.} \{\Xi(t)\}_{t\geq 0}, \quad \lambda \to \infty.$$

Proof. We first note the following representation so called William's formula;

$$A^{-1}(t) = t + T_-\big(T_+^{-1}(t)\big), \quad t \geq 0. \tag{7}$$

This formula can easily be understood in the following way. For given $t > 0$, $A^{-1}(t)$ denotes the necessary time for the occupation time on S_+ of X to reach t. However, in this time interval, the occupation times on S_+ and S_+^c are t and $T_-\big(T_+^{-1}(t)\big)$, respectively. The point is that the first is *not* $T_+\big(T_+^{-1}(t)\big)$. Thus we deduce the formula (7). We refer to [3] for details.

Suppose that functions f and g satisfy

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} L\big(f(\lambda)\big) = a, \quad \lim_{\lambda\to\infty} \frac{1}{\lambda} L\big(g(\lambda)\big) = b.$$

Then, since L is slowly varying, we have from

$$\max\{f(\lambda), g(\lambda)\} \leq f(\lambda) + g(\lambda) \leq 2\max\{f(\lambda), g(\lambda)\}$$

that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} L\big(f(\lambda) + g(\lambda)\big) = \max\{a, b\}.$$

Therefore, by (7) we see that the limit process of

$$\frac{1}{\lambda} L\Big(A^{-1}\big(L^{-1}(\lambda t)\big)\Big)$$

is the larger of t and the limit process of

$$\frac{1}{\lambda} L\Big(T_-\big(T_+^{-1}(L^{-1}(\lambda t))\big)\Big), \quad t \geq 0. \tag{8}$$

Thus, by Lemma 3, we have

$$\frac{1}{\lambda} L\Big(A^{-1}\big(L^{-1}(\lambda t)\big)\Big) \xrightarrow{f.d.} \max\{t, \xi_-(\xi_+^{-1}(t))\}. \tag{9}$$

Notice here that the inverse process of the limit process is $\{\Xi(t)\}_{t\geq 0}$, which is a self-similar process and hence has no fixed discontinuities. Now considering the inverse processes of the both sides of (9), we complete the proof of Theorem 2. □

It remains to prove Theorem 1. However, it is an easy corollary of Theorem 2 and Lemma 1.

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