Robin's inequality and the Riemann hypothesis

By Marek Wójtowicz

Instytut Matematyki, Uniwersytet Kazimierza Wielkiego, Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland

(Communicated by Shigefumi Mori, M.J.A., April 12, 2007)

Abstract: Let $f(n) = \sigma(n)/e^{\gamma} n \log\log n$, $n = 3, 4, \dots$, where σ denotes the sum of divisors function. In 1984 Robin proved that the inequality $f(n) < 1$, for all $n \geq 5041$, is equivalent to the Riemann hypothesis. Here we show that the values of f are close to 0 on a set of asymptotic density 1. Similarly, an inequality by Rosser and Schoenfeld of 1962, dealing with the Euler totient function φ, is essential only on "thin" subsets of $\mathbf{N}$.

Key words: Riemann hypothesis, Robin's inequality, asymptotic density.

1. Introduction. Throughout this paper $\sigma(n)$ and $\varphi(n)$ denote the sum of divisors and the Euler function of n (a positive integer), γ denotes Euler's constant, and $\mathbf{N}$ stands for the set of positive integers. The present note deals with values of the function

$$f(n) = \frac{\sigma(n)}{e^{\gamma} n \log\log n}, \ n \geq 3.$$

In 1984 Robin proved that the Riemann hypothesis (RH) is true if and only if the inequality

$$(R) \qquad f(n) < 1$$

holds for all integers $n \geq 5041$ [11, Théorème 1], and that, independently on RH,

$$(1) \qquad f(n) < 1 + \frac{0.6482\dots}{e^{\gamma}(\log\log n)^2}$$

for all $n \geq 3$, with equality only for $n = 12$ [11, Théorème 2]. It is also known that

$$(2) \qquad \limsup_{n\to\infty} f(n) = 1$$

(see [5, Theorem 323, Sect. 18.3 and 22.9]), and it is obvious that

$$\liminf_{n\to\infty} f(n) = 0$$

(e.g., whenever n runs over prime numbers).

In the context of the two latter equalities it is natural to set the question: *whether the values of f are close to 1 (equivalently, if $\sigma(n) \sim e^{\gamma} n \log\log n$) on some subset $\mathbf{M}$ of $\mathbf{N}$ of positive density*? The main goal of this note is to show this question has a negative answer: roughly speaking, almost all values of f are concentrated around 0, what seems to be somewhat unexpected in the context of Robin's criterion (R) and equality (2).

Theorem 1. *There is a subset $\mathbf{W}$ of $\mathbf{N}$ of asymptotic density 1 such that*

$$(3) \qquad \lim_{\substack{n\to\infty \\ n\in\mathbf{W}}} f(n) = 0.$$

Consequently, for every $D \in (0, 1]$ there is a subset $\mathbf{W}_D$ of $\mathbf{N}$ of asymptotic density 1 with

$$f(n) < D \qquad \text{for all } n \in \mathbf{W}_D.$$

In particular, Theorem 1 implies that inequalities (R) and (1) are essential only on "thin" subsets of $\mathbf{N}$. The theorem completes also the following result by Robin about the behavior of f on some intervals of positive integers (see [11, Proposition 1]): *There is an infinite sequence of very rarely distributed positive integers $C_1 < C_2 < \dots$* (the so called colossally abundant numbers) *such that the maximum of f on every interval $\{C_j \leq n \leq C_{j+1}\}$, $j = 1, 2, \dots$, is attained at C_j or C_{j+1}* (hence every sequence (n_k) giving the equality in (2) can be replaced by a subsequence (C_{j_s})). Notice that the table of all colossally abundant numbers up to 10^{18}, published in 1944 by Alaoglu and Erdös [1, pp. 468-469], contains only 22 elements (more recent results in this direction are published in [2, 3, 6, 10]).

From Theorem 1 we immediately obtain

Corollary 1. *Every subset $\mathbf{M}$ of $\mathbf{N}$ of the asymptotic density $d(\mathbf{M}) > 0$ contains a subset $\mathbf{M}_0$ with $d(\mathbf{M}_0) = d(\mathbf{M})$ such that*

$$\lim_{\substack{n\to\infty \\ n\in\mathbf{M}_0}} f(n) = 0.$$

2000 Mathematics Subject Classiffcation. 11M06, 11N37

The proofs are given in the next section. We recall that the (asymptotic) density $d(A)$ of a subset A of $\mathbf{N}$ is defined as the limit

$$\lim_{x\to\infty}\frac{\#\{n\in A: n\le x\}}{x},$$

if it exists. It is well known that if two subsets A, B of $\mathbf{N}$ possess densities, then $d(A\cup B)=d(A)+d(B)-d(A\cap B)$ whenever $d(A\cup B)$ or $d(A\cap B)$ exists, and that the sets of even (or, odd) and squarefree integers have desities $1/2$ and $6/\pi^2$, respectively (see e.g., [14]).

2. The proofs.

Proof of Theorem 1. We shall prove a slightly stronger result than the main claim. Our proof is a combination of two deep results obtained in 1997 by Ford, and in 2002 by Luca and Pomerance:

(F) *There is a constant* $c_0\in[2,39.4]$ *such that*

$$\frac{\sigma(\varphi(n))}{n}\ge\frac{1}{c_0}$$

for all $n\in\mathbf{N}$ (see [4, Theorem 2]; it is conjectured in [8] that $c_0=2$);

(L-P) *For every* $\varepsilon>0$ *there is a subset* $\mathbf{W}_\varepsilon$ *of* $\mathbf{N}$ *of asymptotic density* 1 *such that*

$$\frac{\sigma(\varphi(n))}{\varphi(n)}<(1+\varepsilon)e^\gamma\log\log\log n$$

for all $n\in\mathbf{W}_\varepsilon$ (see [7; Proof of Theorem 1, inequalities (20) and (36)]).

Let us fix $\varepsilon=1$ and put $\mathbf{W}=\mathbf{W}_1$. Then, by (F) and (L-P), we have

$$(4)\quad \frac{n}{\varphi(n)}=\frac{n}{\sigma(\varphi(n))}\cdot\frac{\sigma(\varphi(n))}{\varphi(n)}\le 2c_0e^\gamma\log\log\log n$$

for all $n\in\mathbf{W}$. Now notice that $\sigma(n)/n<n/\varphi(n)$ for all n's (because, if $n=\prod_{j=1}^s p_j^{\alpha_j}$ is the prime factorization of n into prime factors $p_1<\dots<p_s$ then $\varphi(n)=\prod_{j=1}^s p_j^{\alpha_j-1}(p_j-1)$ and $\sigma(n)=\prod_{j=1}^s(p_j^{\alpha_j+1}-1)(p_j-1)^{-1}$ (see [13, pp. 164 and 230]); now easy calculations give $\frac{\sigma(n)\varphi(n)}{n^2}=\prod_{j=1}^s(1-p_j^{-\alpha_j-1})<1$). Then from (4) we obtain

$$(5)\qquad f(n)<\frac{n}{e^\gamma\varphi(n)\log\log n}\le 2c_0\frac{\log\log\log n}{\log\log n}$$

for all $n\in\mathbf{W}$, and the proof is complete.

Proof of Corollary 1. Let $\mathbf{W}$ be as in Theorem 1. Then the set $\mathbf{M}_0:=\mathbf{M}\cap\mathbf{W}$ possesses the desired property because $d(\mathbf{W}\cup\mathbf{M})$ exists (it equals 1), whence

$$d(\mathbf{M})-d(\mathbf{M}_0)=d(\mathbf{W}\cup\mathbf{M})-d(\mathbf{W})=1-1=0.$$

The proof is complete.

Remark. Inequality in (4) is evidently stronger than the result stated in Theorem 1: it supplements the following inequality obtained in 1962 by Rosser and Schoenfeld [12, Theorem 15]:

$$(6)\qquad \frac{n}{\varphi(n)}\le e^\gamma\cdot\log\log n\cdot\left(1+\frac{2.5}{e^\gamma(\log\log n)^2}\right)$$

for every $n\ge 3$ but $n=2\cdot\ 3\cdot\ldots\cdot 23$, where the constant 2.5 in (6) is replaced by 2.50637.

Inequality (4) complements also the result by Nicolas of 1983 [9, Théorème 1], related to (6), that the inequality

$$\frac{n}{\varphi(n)}>e^\gamma\log\log n$$

holds for infinite number of n's: inequalities of this kind, where the right side is multiplicated by a constant $c\in(0,1]$, cannot hold on sets of positive densities.

References

[1] L. Alaoglu and P. Erdös, On highly composite and similar numbers, Trans. Amer. Math. Soc. **56** (1944), 448–469.

[2] K. Briggs, Abundant numbers and the Riemann hypothesis, Experiment. Math. **15** (2006), no. 2, 251–256.

[3] J. H. Bruinier, Primzahlen, Teilersummen und die Riemannsche Vermutung, Math. Semesterber. **48** (2001), no. 1, 79–92.

[4] K. Ford, An explicit sieve bound and small values of $\sigma(\phi(m))$, Period. Math. Hungar. **43** (2001), no. 1-2, 15–29.

[5] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Fifth edition, Oxford Univ. Press, New York, 1979.

[6] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, Amer. Math. Monthly **109** (2002), no. 6, 534–543.

[7] F. Luca and C. Pomerance, On some problems of Mąkowski-Schinzel and Erdős concerning the arithmetical functions ϕ and σ, Colloq. Math. **92** (2002), no. 1, 111–130.

[8] A. Mąkowski and A. Schinzel, On the functions $\varphi(n)$ and $\sigma(n)$, Colloq. Math. **13** (1964), 95–99.

[9] J.-L. Nicolas, Petites valeurs de la fonction d'Euler, J. Number Theory **17** (1983), no. 3, 375–388.

[10] T. Noe, Sequence A073751, published in: The On-Line Encyclopedia of Integer Sequences, 2002. (Electronic).

[11] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. Pures Appl. (9) **63** (1984), no. 2, 187–213.

[12] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. **6** (1962), 64–94.

[13] W. Sierpiński, *Elementary theory of numbers*, Translated from Polish by A. Hulanicki. Monografie Matematyczne, Państwowe Wydawnictwo Naukowe, Warsaw, 1964.

[14] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Translated from the second French edition (1995) by C. B. Thomas, Cambridge Univ. Press, Cambridge, 1995.