Visible actions on irreducible multiplicity-free spaces

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(Communicated by Shigefumi Mori, M.J.A., Sept. 12, 2007)

Abstract: A holomorphic action of a Lie group G on a complex manifold D is called *strongly visible* if there exist a totally real submanifold S which meets every G-orbit in D and an anti-holomorphic diffeomorphism σ such that $\sigma|_S = \mathrm{id}_S$ and σ preserves every G-orbit. In this paper, we prove that Kac's multiplicity-free space is strongly visible, that is, if $(G_{\mathbf{C}}, V)$ is an irreducible multiplicity-free space of a complex reductive Lie group $G_{\mathbf{C}}$, then a compact real form of $G_{\mathbf{C}}$ acts on V in a strongly visible fashion. Furthermore, we give an explicit description of the choice of a totally real submanifold S and an anti-holomorphic involution σ. This gives an evidence to Kobayashi's conjecture [8, Conjecture 3.2], that is, $\dim_{\mathbf{R}} S$ coincides with the rank of the polynomial representation of $G_{\mathbf{C}}$ on $\mathbf{C}[V]$ in this setting.

Key words: Complex manifold; (strongly) visible action; totally real submanifold; multiplicity-free space; multiplicity-free representation.

Suppose a Lie group G acts holomorphically on a complex manifold D.

Definition 1. This action is *strongly visible* if the following two conditions hold:

(a) There exists a real submanifold S (which we call a *slice*) such that

(V.1) $$D = G \cdot S,$$

that is, S meets every G-orbit in D.

(b) There exists an anti-holomorphic diffeomorphism σ on D such that

(S.1) $$\sigma|_S = \mathrm{id}_S,$$

(S.2) σ preserves each G-orbit in D.

We note that the submanifold S is automatically totally real, that is, $T_xS \cap J_x(T_xS) = \{0\}$ holds for any $x \in S$ by the condition (b).

The notion of (strongly) visible actions has been introduced by Kobayashi in [6] as a basic assumption for the propagation theorem of multiplicity-free property from fibers to the space of holomorphic sections, where the submanifold S plays an important role (see [7,9]). To be more precise, Definition 1 is slightly stronger than the original definition of strongly visible actions in [7,9] in the sense that the original definition of strongly visible actions can be verified locally in a G-invariant open subset.

The notion of (strongly) visible actions is also interesting for its own from geometric viewpoints, as one of the three relevant notions: polar actions in Riemannian geometries (e.g. Podestà–Thorbergsson), coisotropic actions in symplectic geometries (Guillemin–Sternberg, Huckleberry–Wurzbacher), and visible actions in complex geometries (Kobayashi) (see [7, §4] for recent progress in this direction).

Another interesting aspect of (strongly) visible actions is that visible actions naturally bring us to various decomposition theorems of Lie groups and homogeneous spaces. For instance, consider the linear fractional action of $G = SL(2, \mathbf{R})$ on a complex upper half plane $\mathcal{H}_+$. Then the Iwasawa decomposition $G = NAK$ explains that the action of a maximal unipotent subgroup N on $\mathcal{H}_+$ is strongly visible, whereas the Cartan decomposition $G = KAK$ explains that the action of a maximal compact subgroup K on $\mathcal{H}_+$ is also strongly visible (see [7, Figures 5.4.1 (a), (c)]). Conversely, we may expect that new examples of (strongly) visible actions could reveal new decomposition theorems of Lie groups and homogeneous spaces. In this sense, finding an explicit description of a totally real submanifold S in Definition 1 becomes a guiding principle to find decomposition theorems.

2000 Mathematics Subject Classification. Primary 32M05; Secondary 22E46, 20G05, 32M15.

Originally, the notion of (strongly) visible actions was introduced to give an unified explanation of various multiplicity-free theorems [6,7]. Conversely if we are given multiplicity-free representations, we could expect that (some real forms of) the group acts on the underlying geometry in a (strongly) visible fashion (e.g. [10,11]).

In this paper, we consider multiplicity-free spaces in the sense of Kac [4]. Given an algebraic representation $\pi : G_{\mathbf{C}} \to GL_{\mathbf{C}}(V)$ of a connected complex reductive Lie group $G_{\mathbf{C}}$ on a finite dimensional complex vector space V, we have a representation of $G_{\mathbf{C}}$ on the polynomial ring $\mathbf{C}[V]$. We say V is a *mutiplicity-free space of* $G_{\mathbf{C}}$ (or simply, $(G_{\mathbf{C}}, V)$ is a multiplicity-free space) if $\mathbf{C}[V]$ decomposes into the multiplicity-free sum of irreducible representations of $G_{\mathbf{C}}$.

We write the irreducible decomposition of $\mathbf{C}[V]$ as follows:

$$\mathbf{C}[V] \simeq \bigoplus_{\lambda \in \Lambda} P_\lambda.$$

Here, P_λ is an irreducible representation of $G_{\mathbf{C}}$ with a highest weight λ. Then the set of highest weights Λ forms a semigroup, and there exist linearly independent highest weights $\lambda_1, \ldots, \lambda_k \in \Lambda$ such that $\Lambda = \{r_1\lambda_1 + \cdots + r_k\lambda_k : r_1, \ldots, r_k \in \mathbf{Z}_{\geq 0}\}$ ([2,5]). We say the rank k of the semigroup Λ is the *rank* of the polynomial representation $\mathbf{C}[V]$ (or the number of fundamental generators), which was computed in Howe and Umeda [3, Table 15.1].

The multiplcity-free space $(G_{\mathbf{C}}, V)$ is *irreducible* if π is irreducible. The classification of irreducible multiplicity-free spaces was accomplished by Kac [4, Theorem 3] under the assumption that $G_{\mathbf{C}}$ is reductive (see also [2] for a survey). According to the classification, $G_{\mathbf{C}}$ is of the form $H_{\mathbf{C}} \times \mathbf{C}^*$ or $H_{\mathbf{C}}$ where $H_{\mathbf{C}}$ is a semisimple complex Lie group. We recall his list in a way that fits into our framework of visible actions.

Since $G_{\mathbf{C}}$ is reductive, there exists a compact real form of $G_{\mathbf{C}}$, denoted by G_U. By Weyl's unitary trick, the category of holomorphic representations $(G_{\mathbf{C}}, V)$ is equivalent to that of complex representations (G_U, V). So we give Kac's classification of irreducible multiplicity-free spaces in terms of (G_U, V) in Table I. Here are some conditions on integers m and n in Table I. In (1a) $n \geq 2$; in (3) $n \geq 3$; in (6a) $m \neq n$; in (7) $n \geq 2$; in (8) $n \geq 2$; in (9a) $m \geq 5$. In (13) and (14), G_2 and E_6 denote the corresponding simply connected Lie groups of exceptional type. Since local isomorphisms of G_U are not the main issue here, we shall use the global form of G_U as in Table I.

Table I. Irreducible multiplicity-free spaces

	Kac's list		Slice
	G_U	V	$\dim_{\mathbf{R}} S$
1a	$SU(n)$	$\mathbf{C}^n$	1
1b	$\mathbf{T}$	$\mathbf{C}$	
2	$Sp(n)$	$\mathbf{C}^{2n}$	1
3	$SO(n) \times \mathbf{T}$	$\mathbf{C}^n$	2
4	$SU(n) \times \mathbf{T}$	$S^2(\mathbf{C}^n)$	n
5a	$SU(2n+1)$	$\bigwedge^2(\mathbf{C}^{2n+1})$	n
5b	$SU(2n) \times \mathbf{T}$	$\bigwedge^2(\mathbf{C}^{2n})$	
6a	$SU(m) \times SU(n)$	$\mathbf{C}^m \otimes \mathbf{C}^n$	$\min(m,n)$
6b	$SU(n) \times SU(n) \times \mathbf{T}$	$\mathbf{C}^n \otimes \mathbf{C}^n$	
7	$SU(2) \times Sp(n) \times \mathbf{T}$	$\mathbf{C}^2 \otimes \mathbf{C}^{2n}$	3
8	$SU(3) \times Sp(n) \times \mathbf{T}$	$\mathbf{C}^3 \otimes \mathbf{C}^{2n}$	5 ($n=2$) 6 ($n \geq 3$)
9a	$SU(m) \times Sp(2)$	$\mathbf{C}^m \otimes \mathbf{C}^4$	6
9b	$SU(4) \times Sp(2) \times \mathbf{T}$	$\mathbf{C}^4 \otimes \mathbf{C}^4$	
10	$Spin(7) \times \mathbf{T}$	$\mathbf{C}^8$	2
11	$Spin(9) \times \mathbf{T}$	$\mathbf{C}^{16}$	3
12	$Spin(10)$	$\mathbf{C}^{16}$	2
13	$G_2 \times \mathbf{T}$	$\mathbf{C}^7$	2
14	$E_6 \times \mathbf{T}$	$\mathbf{C}^{27}$	3

Remark 2. In (1a), (2), (5a), (6a), (9a) and (12), G_U is semisimple. We note that $(G_U \times \mathbf{T}, V)$ also give irreducible multiplicity-free spaces in these cases (see [4, Theorem 3]) but we have omitted there trivial cases in Table I.

Our main theorem is stated as follows:

Theorem A. *Let $(G_{\mathbf{C}}, V)$ be an irreducible multiplicity-free space, and G_U a compact real form of $G_{\mathbf{C}}$.*

(1) *The action of G_U on V is strongly visible.*
(2) *We can take a slice S (see Definition 1) such that $\dim_{\mathbf{R}} S$ is equal to the rank of the polynomial representation $\mathbf{C}[V]$.*

The dimension of our slice S is listed in the right column of Table I.

Combining Theorem A with [7, Theorem 5] (or [9, Corollary 2.4]), we obtain the following corollary immediately.

Corollary B. *Let $\pi : G_{\mathbf{C}} \to GL_{\mathbf{C}}(V)$ be an irreducible algebraic representation of a connected complex reductive Lie group $G_{\mathbf{C}}$ on a finite dimensional complex vector space V. Then the following* *two conditions are equivalent:*

(i) *V is a multiplicity-free space of $G_{\mathbf{C}}$.*

(ii) *The action of a compact real form G_U of $G_{\mathbf{C}}$ on V is strongly visible in the sense of Definition* 1.

The rest of this paper is devoted to the proof of Theorem A. For the cases discussed in Remark 2, the orbit of $G_U \times \mathbf{T}$ on V coincides with that of G_U on V. Therefore Theorem A for $G_U \times \mathbf{T}$ implies Theorem A for G_U and vice versa. Thus, we may and do assume that G_U always contains one dimensional center $\mathbf{T}$. Now we divide irreducible multiplicity-free spaces into the following three cases.

Case 1. (G_U, V) is in cases (1), (3), (4), (5), (6), (12) and (14).

Case 2. (G_U, V) is in cases (7), (8) and (9), that is, of the form $(SU(m) \times Sp(n) \times \mathbf{T}, \mathbf{C}^m \otimes \mathbf{C}^{2n})$.

Case 3. We divide the remaining cases (2), (10), (11) and (13) into the following subcases.

(3-a) $(G_U, V) = (Sp(n) \times \mathbf{T}, \mathbf{C}^{2n})$.

(3-b) $(G_U, V) = (Spin(7) \times \mathbf{T}, \mathbf{C}^8)$ and $(G_2 \times \mathbf{T}, \mathbf{C}^7)$.

(3-c) $(G_U, V) = (Spin(9) \times \mathbf{T}, \mathbf{C}^{16})$.

First, we consider Case 1.

Proof of Theorem A in Case 1. We shall see below that the proof of Theorem A in this case reduces to a special case of [11, Theorem 1.5].

Suppose (G_U, V) is in Case 1. Then there exists a non-compact, simply connected simple Lie group G of Hermitian type such that a maximal compact subgroup K of G is isomorphic to G_U and the induced action of the adjoint representation $(K, \mathfrak{g}/\mathfrak{k})$ is isomorphic to the given representation (G_U, V) (up to the action of the center of G_U). Here $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ is that of K. On the other hand, it is proved in [11, Theorem 1.5] that the K-action on the Hermitian symmetric space G/K is strongly visible. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of Lie algebra $\mathfrak{g}$. Since K acts on $\mathfrak{p}$ as an adjoint representation and G/K is realized as a bounded symmetric domain in $\mathfrak{p}$, we can show similarly that the K-action on $\mathfrak{p}$ is strongly visible with a choice of slice S such that S is a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. In particular $\dim_{\mathbf{R}} S = \dim \mathfrak{a} = \mathbf{R}\text{-rank}\, G/K$. $\square$

We will call (G_U, V) in Case 1 of "Hermitian type".

Next, we consider Case 2. We identify $SU(m) \times Sp(n) \times \mathbf{T} \simeq U(m) \times Sp(n)$ and $\mathbf{C}^m \otimes \mathbf{C}^{2n} \simeq M(m, 2n; \mathbf{C})$, the vector space consisting of $(m \times 2n)$-complex matrices. Here we let $U(m) \times Sp(n)$ act on $M(m, 2n; \mathbf{C})$ by

$$(g, h) \cdot X = gXh^{-1} \ (g \in U(m),\ h \in Sp(n))$$

where we realize $Sp(n)$ in $U(2n)$ in a standard way.

Proof of Theorem A in Case 2. Case 2 consists of three cases (7), (8) and (9), for which the proof of Theorem A can be given similarly. Here, we consider only the case (7), namely, $(G_U, V) = (U(2) \times Sp(n), M(2, 2n; \mathbf{C}))$, which is simplest among the three cases. Let $\vec{e}_1, \dots, \vec{e}_{2n}$ be the standard basis of $\mathbf{C}^{2n}$. We write an element $X \in V$ as

$$X = \begin{pmatrix} x_1 & \cdots & x_{2n} \\ y_1 & \cdots & y_{2n} \end{pmatrix} = \begin{pmatrix} {}^t\vec{x} \\ {}^t\vec{y} \end{pmatrix}.$$

By using the first factor $U(2)$ of G_U, we can transform X such that $(\vec{x}, \vec{y}) = 0$ where $(\,\cdot\,,\,\cdot\,)$ denotes the standard Hermitian inner product on $\mathbf{C}^{2n}$.

Second, we take $h_1 \in Sp(n)$ such that ${}^t\vec{x}h_1^{-1} = r_1 {}^t\vec{e}_1$ for $r_1 = (|x_1|^2 + \cdots + |x_{2n}|^2)^{1/2}$. Let

$$\begin{pmatrix} {}^t\vec{x'} \\ {}^t\vec{y'} \end{pmatrix} := \begin{pmatrix} {}^t\vec{x}h_1^{-1} \\ {}^t\vec{y}h_1^{-1} \end{pmatrix} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ y'_1 & y'_2 & \cdots & y'_{2n} \end{pmatrix}.$$

Since $(\vec{x'}, \vec{y'}) = (\vec{x}, \vec{y}) = 0$, we get $y'_1 = 0$. Thus Xh_1^{-1} is of the form

$$Xh_1^{-1} = \begin{pmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & y'_2 & y'_3 & \cdots & y'_{2n} \end{pmatrix}.$$

Third, we write

$$y'_2 = r_2 e^{\sqrt{-1}\theta} \ (r_2 \geq 0,\ \theta \in \mathbf{R}/2\pi\mathbf{Z})$$

and take $h_2 \in Sp(n-1)$ such that

$$e^{-\sqrt{-1}\theta}(y'_3, \dots, y'_{2n})h_2^{-1} = r_3(1, 0, \dots, 0)$$

for $r_3 = (|y'_3|^2 + \cdots + |y'_{2n}|^2)^{1/2}$. We regard $Sp(n-1)$ as a subgroup of $Sp(n)$ and set $g_1 := \mathrm{diag}(1, e^{-\sqrt{-1}\theta}) \in U(2)$. Then

$$g_1(Xh_1^{-1})h_2^{-1} = \begin{pmatrix} r_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & r_3 & 0 & \cdots & 0 \end{pmatrix} =: X(r_1, r_2, r_3).$$

This implies that

$$S = \{X(r_1, r_2, r_3) : r_1, r_2, r_3 \in \mathbf{R}\}$$

meets every G_U-orbit in V ($S_+ := \{X(r_1, r_2, r_3) : r_1, r_2, r_3 \geq 0\}$ already holds this property). Now we define an anti-holomorphic involution on $M(2, 2n; \mathbf{C})$ by $\sigma(X) = \overline{X}$ ($X \in M(2, 2n; \mathbf{C})$). Then it is clear that $\sigma|_S = \mathrm{id}_S$ and σ preserves each

$(U(2) \times Sp(n))$-orbit in $M(2, 2n; \mathbf{C})$. Therefore this action is strongly visible with the data (S, σ). $\square$

Finally, we consider Case 3. Let W be a real vector space equipped with an inner product $(\cdot, \cdot)_W$. We write $S(W)$ for the unit sphere in W. Here is an elementary observation:

Lemma 3. *If a group G acts linearly on W and acts transitively on $S(W)$, then*

$$W = G \cdot \mathbf{R}v_0$$

for any $v_0 \in S(W)$.

In addition to Lemma 3, we prepare some more lemmas.

Lemma 4. *Retain the setting of Lemma 3. Let $W_1 = (\mathbf{R}v_0)^\perp$ be the orthogonal complementary subspace of $\mathbf{R}v_0$ in W We denote by G_{v_0} the isotropy subgroup of G at v_0. If G_{v_0} acts transitively on $S(W_1)$, then we have*

$$W \oplus W = G \cdot (\mathbf{R}v_0 \oplus (\mathbf{R}v_0 \oplus \mathbf{R}v_1))$$

for any $v_1 \in S(W_1)$. Here the G-action on $W \oplus W$ is given by the diagonal action; $g \cdot (w_1, w_2) = (gw_1, gw_2)$ $(g \in G)$.

Proof. By the assumption that G_{v_0} acts transitively on $S(W_1)$, we obtain $W_1 = G_{v_0} \cdot \mathbf{R}v_1$ for any $v_1 \in S(W_1)$ from Lemma 3. In view of the decomposition $W = \mathbf{R}v_0 \oplus W_1$, we have

$$\begin{aligned} W &= \mathbf{R}v_0 \oplus (G_{v_0} \cdot \mathbf{R}v_1) \\ &= G_{v_0} \cdot (\mathbf{R}v_0 \oplus \mathbf{R}v_1) \end{aligned}$$

Hence, Lemma 4 is proved. $\square$

Lemma 4 may be regarded as a special case of the following lemma ($W_3 = \{0\}$).

Lemma 5. *Retain the setting of Lemma 3 and we write $W_1 = (\mathbf{R}v_0)^\perp$. Assume further that W_1 decomposes as a direct sum of two vector subspaces W_2 and W_3 with the following properties*:

(a) *G_{v_0} acts transitively on $S(W_2)$.*

(b) *For a fixed $v_1 \in S(W_2)$, $G_{v_0,v_1} := G_{v_0} \cap G_{v_1}$ acts transitively on $S(W_3)$.*

Then we have

$$W \oplus W = G \cdot (\mathbf{R}v_0 \oplus (\mathbf{R}v_0 \oplus \mathbf{R}v_1 \oplus \mathbf{R}v_2))$$

for any $v_2 \in S(W_3)$.

Proof. The proof is similar to that of Lemma 4. $\square$

$Spin(2n+1)$ acts on $\mathbf{R}^{2^n}$ as a spin representation. G_2 is an automorphism group $\mathrm{Aut}_{\mathbf{R}}(\mathfrak{C})$, where $\mathfrak{C} \simeq \mathbf{R}^8$ stands for the Cayley algebra. Let $e_0, \dots, e_7$ be the standard basis of $\mathfrak{C}$, and we set a subspace $\mathfrak{C}_0 := \{x_1e_1 + \cdots + x_7e_7 : x_1, \dots x_7 \in \mathbf{R}\} \simeq \mathbf{R}^7$. Then $\mathfrak{C}_0$ is G_2-invariant, so we define a representation of G_2 on $\mathfrak{C}_0$ (see [1], for example).

Table II. Transitive G_U-actions on $S(W)$

	G	W	$S(W)$	G_v
1	$Sp(n)$	$\mathbf{C}^{2n} \simeq \mathbf{R}^{4n}$	S^{4n-1}	$Sp(n-1)$
2	$Spin(9)$	$\mathbf{R}^{16}$	S^{15}	$Spin(7)$
3	$Spin(7)$	$\mathbf{R}^8$	S^7	G_2
4	G_2	$\mathbf{R}^7$	S^6	$SU(3)$
5	$SU(3)$	$\mathbf{C}^3 \simeq \mathbf{R}^6$	S^5	$SU(2)$

Now we recall,

Fact 6. *The representation of G on a real vector space W in Table II induces the transitive action on the unit sphere $S(W)$.*

The isomorphic class of isotropy subgroups G_v is listed in the right column in Table II.

We are ready to prove Theorem A in Case 3.

Proof of Theorem A in Case 3-a. It follows from Fact 6 (1) and Lemma 3 that we have

$$\mathbf{C}^{2n} = Sp(n) \cdot \mathbf{R}v_0$$

for any $v_0 \in S^{4n-1}$. This means that the totally real submanifold $S := \mathbf{R}v_0$ meets every $Sp(n)$-orbit in $\mathbf{C}^{2n}$. As before, let $\vec{e}_1, \dots, \vec{e}_{2n}$ be the standard basis of $\mathbf{C}^{2n}$. Now we take $v_0 = \vec{e}_1$. Let σ be the standard complex conjugate of $\mathbf{C}^{2n}$. Then, clearly, σ preserves every $Sp(n)$-orbit in $\mathbf{C}^{2n}$. With the data (S, σ) the natural action of $Sp(n)$ on $\mathbf{C}^{2n}$ is strongly visible. $\square$

Remark 7. The strongly visibility of $SU(n)$ acting on $\mathbf{C}^n$ is already proved in Case 1. An analogous idea to the previous proof of Case 3-a gives an alternative proof of the strongly visibility because $SU(n)$ acts on S^{2n-1} transitively in $n \geq 2$.

Proof of Theorem A in Case 3-b. We first consider the case $(G_U, V) = (G_2 \times \mathbf{T}, \mathbf{C}^7)$. The idea of the proof is similar to Case 3-a, but we need to iterate the argument twice as follows:

Let $G_2(\mathbf{C})$ be the complexification of G_2, and $\mathbf{C}^7 = \mathbf{R}^7 + \sqrt{-1}\mathbf{R}^7 \simeq \mathfrak{C}_0 \otimes_{\mathbf{R}} \mathbf{C}$.

From Fact 6 (iv) and Lemma 3, we have

$$\mathbf{R}^7 = G_2 \cdot \mathbf{R}v_0$$

for any $v_0 \in S^6$. Since the isotropy subgroup $(G_2)_{v_0} \simeq SU(3)$ acts transitively on the complementary subspace $(\mathbf{R}v_0)^\perp \simeq \mathbf{R}^6$, it follows from Lemma 4 that we have

$$\mathbf{C}^7 = G_2 \cdot (\mathbf{R}v_0 + \sqrt{-1}(\mathbf{R}v_0 + \mathbf{R}v_1))$$

for any $v_1 \in S^5$. Finally, by using the action of the second factor $\mathbf{T}$ of G_U on $\mathbf{R}^7$, we get

$$\mathbf{C}^7 = (G_2 \times \mathbf{T}) \cdot (\mathbf{R}v_0 + \sqrt{-1}\mathbf{R}v_1).$$

Thus

$$S := \mathbf{R}v_0 + \sqrt{-1}\mathbf{R}v_1 \simeq \mathbf{R}^2$$

meets every G_U-orbit in $\mathbf{C}^7$.

We recall $e_1, \ldots, e_7$ is a basis of $\mathfrak{C}_0$, and take $v_0 = e_1,\ v_1 = e_2$. We define an anti-holomorphic involution σ in $\mathbf{C}^7 \simeq \mathfrak{C}_0 \otimes_{\mathbf{R}} \mathbf{C}$ by

$$\sigma(c_1e_1 + \cdots + c_7e_7) = \overline{c}_1e_1 - (\overline{c}_2e_2 + \cdots + \overline{c}_7e_7)$$

for $c_1, \ldots, c_7 \in \mathbf{C}$. Then this σ preserves each $(G_2 \times \mathbf{T})$-orbit in $\mathbf{C}^7$. Therefore, this action is strongly visible with the data (S, σ).

The remaining case of Case 3-b is $(G_U, V) = (\mathit{Spin}(7), \mathbf{C}^8)$. As G_2 and its subgroup $SU(3)$ act transitively on S^6 and S^5, respectively, in the previous case, we use a fact that $\mathit{Spin}(7)$ and its subgroup G_2 act transitively on S^7 and S^6, respectively (see Fact 6), in this case. Since the proof parallels to the previous case, we omit its proof. □

Proof of Theorem A in Case 3-c. This case treats $(G_U, V) = (\mathit{Spin}(9), \mathbf{C}^{16})$.

The proof in this case is similar to Case 3-b. We apply Lemma 5 in place of Lemma 4. The key ingredient of the proof is that the triple of Lie groups

$$\mathit{Spin}(9) \supset \mathit{Spin}(7) \supset G_2$$

act transitively on the triple of unit spheres

$$S^{15} \supset S^7 \supset S^6,$$

respectively. Correspondingly, we can take S to be a three dimensional real vector subspace $\mathbf{R}^3$ in $\mathbf{C}^{16}$ such that

$$\mathbf{C}^{16} = (\mathit{Spin}(9) \times \mathbf{T}) \cdot \mathbf{R}^3.$$

We omit details. □

Remark 8. We note that key subgroups in the proof of Cases 3-b and 3-c give rise to non-symmetric spherical varieties

$$G_2/SU(3),\ \mathit{Spin}(7)/G_2,\ \mathit{Spin}(9)/\mathit{Spin}(7).$$

Finally, the second statement of Theorem A follows by comparing the list of $\dim_{\mathbf{R}} S$ for slices S (see the right column in Table I) with the list of the rank of polynomial representations $\mathbf{C}[V]$ (the number of fundamental generators) given in [3, Table 15.1].

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