The best constant of Sobolev inequality in an n dimensional Euclidean space

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(Communicated by Heisuke HIRONAKA, M. J. A., March 14, 2005)

Abstract: The best constant of Sobolev inequality in an n dimensional Euclidean space is found by means of the theory of reproducing kernel and Green function. The concrete form of the best constant is also found in the case of Sobolev space $W^2(\mathbf{R}^n)$ $(n = 2, 3)$.

Key words: Best constant; Sobolev inequality; reproducing kernel; Green function.

1. Introduction. Let $H := W^M(\mathbf{R}^n)$ be the Sobolev space of order M satisfying $2M > n$, then the well-known Sobolev's inequality (Sobolev embedding) [1, Cor. 9.13] asserts

$$\left(\sup_{y\in\mathbf{R}^n} |u(y)|\right)^2 \le C\|u\|_H^2, \tag{1.1}$$

where $\|\cdot\|_H$ is the norm of H which is induced by the certain inner product attached to H. We here adopt the following generalized inner product.

$$\begin{aligned}
(u,v)_H &:= \int_{\mathbf{R}^n}\Bigg[\sum_{j=0}^{[M/2]} p_{M-2j}\left(\Delta^j u(x)\right)\overline{\left(\Delta^j v(x)\right)} \\
&\qquad + \sum_{j=0}^{[\frac{M-1}{2}]} p_{M-2j-1}\nabla\left(\Delta^j u(x)\right)\cdot\overline{\nabla\left(\Delta^j v(x)\right)}\Bigg]dx \\
&= (2\pi)^{-n}\int_{\mathbf{R}^n}(-1)^M\, p(-|\xi|^2)\,\widehat{u}(\xi)\,\overline{\widehat{v}(\xi)}\,d\xi \\
&\qquad (\forall u, v\in H),
\end{aligned} \tag{1.2}$$

where $p(\lambda) = \prod_{j=0}^{M-1}(\lambda-\alpha_j) = \sum_{j=0}^{M}(-1)^j\,p_j\,\lambda^{M-j}$ and $\{\alpha_i\}_{i=0}^{M-1}$ are positive numbers satisfying $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_{M-1}$. The purpose of this paper is to find the best constant of the Sobolev inequality. It should be noted that Morosi and Pizzocchero [3] obtained the best constant in the the degenerated case $(\alpha_0 = \cdots = \alpha_{M-1} = 1)$.

Details of this result are published in [2].

2. Main results. In this section, we present the main theorems of this paper. Before going to the main theorems, we start with computing the reproducing kernel of H which is needed for the proof of the main Theorem 2.3. Let $G(\alpha; x)$ be the Green function of the differential operator $(-1)^M p(\Delta)$. Then the following theorem holds:

Theorem 2.1. *Assume $2M > n$ then Green function $K(x,y) = G(\alpha; x-y)$ is the reproducing kernel with respect to the Hilbert space H and an inner product $(u,v)_H$ of Eq.* (1.2)*. That is to say, for any $y\in\mathbf{R}^n$ $K(x,y)$ belongs to H as a function of x. For almost all $y\in\mathbf{R}^n$ we have*

$$(u(x), K(x,y))_H = u(y).$$

Proof of Theorem 2.1. Since

$$\widehat{G}(\alpha;\xi) = (-1)^M p(-|\xi|^2)^{-1}, \tag{2.1}$$

$\widehat{K}(\xi,y) = e^{-\sqrt{-1}\langle\xi,y\rangle}\widehat{G}(\alpha;\xi)$ holds. In order to show that $K(x,y)\in H$, it is enough to show that $|\xi|^M|\widehat{G}(\alpha;\xi)|\in L^2(\mathbf{R}^n)$, but this is assured by the condition $n < 2M$. By Eq. (1.2) we have

$$\begin{aligned}
&(u(x), K(x,y))_H \\
&= (2\pi)^{-n}\int_{\mathbf{R}^n}\Big\{(-1)^M\,p(-|\xi|^2)\,\widehat{u}(\xi) \\
&\qquad\qquad \cdot\overline{e^{-\sqrt{-1}\langle\xi,y\rangle}\,\widehat{G}(\alpha;\xi)}\Big\}\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbf{R}^n} e^{\sqrt{-1}\langle\xi,y\rangle}\,\widehat{u}(\xi)\,d\xi = u(y)
\end{aligned}$$

where we have used (2.1). This completes the proof of Theorem 2.1. □

2000 Mathematics Subject Classification. Primary 46E35, 46E22.

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Using the well-known theory of reproducing kernel, we have the main results of this paper.

Theorem 2.2. *Let $2M > n$, then for every u in H there exists a positive constant C which is independent of u such that the following Sobolev inequality holds.*

$$\left(\sup_{y\in\mathbf{R}^n} |u(y)|\right)^2 \leq C\|u\|_H^2 = C(u,u)_H. \tag{2.2}$$

Among these constants C the best constant is

$$C(n,M) = \sup_{y\in\mathbf{R}^n} K(y,y) = G(\alpha;0). \tag{2.3}$$

If we replace C by $C(n,M)$ in the above inequality, the equality holds for $u(x) = K(x,y)$ for every fixed $y \in \mathbf{R}^n$.

The proof of this theorem is easy so we omit it; see [4, p. 812]. The best constants are given by the following main theorem.

Theorem 2.3. (1) *For $M = 2,3,4,\ldots,$ we assume that an odd number $n = 2q+2$ satisfies $q - 1/2 = 0,1,2,\ldots,M-2$. Then we have*

$$\begin{aligned} &C(2q+2,M) \\ &= (-1)^{M+q-1/2}\,\frac{1}{4\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^q \sum_{j=0}^{M-1} e_j\,\alpha_j^q \\ &= (-1)^{M+q-1/2}\,\frac{1}{4\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^q \\ &\quad\cdot \left|\begin{array}{c} \alpha_j^i \\ \hline \alpha_j^q \end{array}\right| \Big/ \left|\ \alpha_j^i\ \right| \end{aligned} \tag{2.4}$$

where the numerator is the determinant of an $M \times M$ matrix $(0 \leq i \leq M-2,\ 0 \leq j \leq M-1)$ and the denominator is the determinant of an $M \times M$ Vandermonde matrix and $e_j = 1/p'(\alpha_j)$ $(0 \leq j \leq M-1)$.

(2) *For $M = 2,3,4,\ldots$ we assume that an even number $n = 2q+2$ satisfies $q = 0,1,2,\ldots,M-2$. Then we have*

$$\begin{aligned} &C(2q+2,M) \\ &= (-1)^{M+q}\,\frac{1}{\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^{q+1} \sum_{j=0}^{M-1} e_j\,\alpha_j^q\,\log\alpha_j \\ &= (-1)^{M+q-1/2}\,\frac{1}{4\Gamma(q+1)}\left(\frac{1}{4\pi}\right)^q \\ &\quad\cdot \left|\begin{array}{c} \alpha_j^i \\ \hline \alpha_j^q\log\alpha_j \end{array}\right| \Big/ \left|\ \alpha_j^i\ \right|. \end{aligned} \tag{2.5}$$

Lemma 2.1. *The Green function for the differential operator $(-1)^M\,p(\Delta)$ has the following integral representation:*

$$\begin{aligned} G(\alpha;x) &= \int_0^\infty e(\alpha;t)\,H(x,t)\,dt, \\ e(\alpha;t) &= (-1)^{M-1}\sum_{j=0}^{M-1} e_j\,e^{-\alpha_j t}, \\ H(x,t) &= (4\pi t)^{-n/2}\,\exp(-|x|^2/(4t)), \end{aligned} \tag{2.6}$$

where $H(x,t)$ is the heat kernel.

Proof of Lemma 2.1. Applying the expansion formula by partial fractions

$$p(\lambda)^{-1} = \sum_{j=0}^{M-1} e_j\,(\lambda-\alpha_j)^{-1}$$

to Eq. (2.1), we have

$$\begin{aligned} \widehat{G}(\alpha;\xi) &= (-1)^M\sum_{j=0}^{M-1} e_j\,(-|\xi|^2-\alpha_j)^{-1} \\ &= (-1)^{M-1}\int_0^\infty \sum_{j=0}^{M-1} e_j e^{-(|\xi|^2+\alpha_j)t}dt \\ &= \int_0^\infty e(\alpha;t)\,e^{-|\xi|^2 t}\,dt. \end{aligned}$$

Using well known formula $\widehat{H}(\xi,t) = e^{-|\xi|^2 t}$ we obtain (2.6). □

Now we state a fundamental lemma concerning $e(\alpha;t)$.

Lemma 2.2. *$e(\alpha;t)$ is an entire function of t and can be expressed by a Taylor series*

$$e(\alpha;t) = (-1)^{M-1}\sum_{i=M-1}^{\infty}\left(\sum_{j=0}^{M-1}\alpha_j^i e_j\right)\frac{(-1)^i}{i!}\,t^i. \tag{2.7}$$

This follows at once from the well known fact

$$\sum_{j=0}^{M-1}\alpha_j^i\,e_j = \delta_{i,M-1} = \begin{cases} 0 & (0 \leq i \leq M-2), \\ 1 & (i = M-1). \end{cases} \tag{2.8}$$

Before going to the proof of Theorem 2.3, we prove another important fact.

Proposition 2.1. *$G(\alpha;\cdot)$ is positive.*

Proof of Proposition 2.1. It is enough to show that $e(\alpha;t) > 0$ from (2.6). From (2.6) and (2.8), we see that $e(\alpha;t) = e(\alpha_0, \alpha_1, \dots, \alpha_{M-1};t)$ is the solution of the initial value problem:

(2.9)

$$\Big(\frac{d}{dt} + \alpha_{M-1}\Big)\cdots\Big(\frac{d}{dt} + \alpha_1\Big)\Big(\frac{d}{dt} + \alpha_0\Big)e(\alpha;t) = 0,$$

$$e^{(j)}(\alpha;0) = 0 \ (0 \le i \le M-2), \quad 1 \ (i = M-1).$$

From this fact, we know that $e(\alpha;t)$ can be expressed as follows:

$$(2.10) \qquad e(\alpha;t) = e(\alpha_0, \alpha_1, \dots, \alpha_{M-1};t) = (e(\alpha_0;\cdot) * \cdots * e(\alpha_{M-1};\cdot))(t).$$

Since $e(\alpha_0;t) = e^{-\alpha_0 t} > 0$, by induction, $e(\alpha;t)$ is positive. □

Proof of Theorem 2.3. By Theorem 2.2 we have

$$C(2q+2, M) = G(\alpha;0) = \int_0^\infty e(\alpha;t)\,H(0;t)\,dt = \Big(\frac{1}{4\pi}\Big)^{q+1}\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt.$$

We assumed $n = 2q+2 > 2M$ that is $M-q-1 > 0$ so the above integral is convergent due to the Taylor expansion $e(\alpha;t) = t^{M-1}/(M-1)! + \cdots$. At first we treat the case (1). Integrating the relation

$$\Bigg[\sum_{j=0}^{q-1/2} e^{(j)}(\alpha;t)\,\Gamma(q-j)\,t^{-(q-j)}\Bigg]' =$$

$$e^{(q+1/2)}(\alpha;t)\,\Gamma(1/2)\,t^{-1/2} - e(\alpha;t)\,\Gamma(q+1)\,t^{-(q+1)}$$

on the interval $0 < t < \infty$ we obtain

$$0 = \Gamma(1/2)\int_0^\infty e^{(q+1/2)}(\alpha;t)\,t^{-1/2}\,dt - \Gamma(q+1)\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt.$$

Hence we have

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = (-1)^{M+q-1/2}\,\frac{\Gamma(1/2)^2}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^q.$$

This proves case (1).

Next we treat the case (2). Integrating the relation

$$\Bigg[\sum_{j=0}^{q-1} e^{(j)}(\alpha;t)\,\Gamma(q-j)\,t^{-(q-j)}\Bigg]' = e^{(q)}(\alpha;t)t^{-1} - e(\alpha;t)\,\Gamma(q+1)\,t^{-q-1}$$

on the interval $0 < t < \infty$ we obtain

$$\begin{aligned} 0 &= \Bigg[\sum_{j=0}^{q-1} e^{(j)}(\alpha;t)\,\Gamma(q-j)\,t^{-(q-j)} - e^{(q)}(\alpha;t)\,\log t\Bigg]\Bigg|_{t=0}^{t=\infty} \\ &= -\Gamma(q+1)\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt - \int_0^\infty e^{(q+1)}(\alpha;t)\,\log t\,dt. \end{aligned}$$

So we have

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = \frac{(-1)^{M+q-1}}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^{q+1}\int_0^\infty e^{-\alpha_j t}\,\log t\,dt.$$

Considering that $\int_0^\infty e^{-\alpha_j t}\,\log t\,dt = -\alpha_j^{-1}(\log\alpha_j + \gamma)$ where $\gamma = 0.577\cdots$ is the constant of Euler, we finally obtain

$$\int_0^\infty e(\alpha;t)\,t^{-q-1}\,dt = \frac{(-1)^{M+q}}{\Gamma(q+1)}\sum_{j=0}^{M-1} e_j\,\alpha_j^q\,\log\alpha_j.$$

This completes the proof of case (2). □

3. Special case. We here treat the simplest case $M = 2$, where the condition $n < 2M$ means that $n = 2, 3$. The Sobolev space $H = W^2(\mathbf{R}^n)$ consists of all the functions $u(x) \in L^2(\mathbf{R}^n)$ satisfying $\Delta u(x) \in L^2(\mathbf{R}^n)$. The inner product for any u, v in H is given by

$$(u,v)_H = \int_{\mathbf{R}^n}\big[(\Delta u(x))\,\overline{(\Delta v(x))} + p_1\,(\nabla u(x))\cdot\overline{(\nabla v(x))} + p_2\,u(x)\,\overline{v(x)}\,\big]dx$$

where $p_1 = \alpha_0 + \alpha_1$, $p_2 = \alpha_0\alpha_1$ $(0 < \alpha_0 < \alpha_1)$. As a special case of Theorem 2.2 and 2.3 we have

Theorem 3.1. *For every $u(x)$ in $H = W^2(\mathbf{R}^n)$ $(n = 2, 3)$ there exists a positive constant C independent of $u(x)$ such that the following Sobolev inequality holds.*

(3.1)

$$\Big(\sup_{y\in\mathbf{R}^n}|u(y)|\Big)^2 \le C\int_{\mathbf{R}^n}(|\Delta u(x)|^2 + p_1\,|\nabla u(x)|^2 + p_2\,|u(x)|^2)\,dx.$$

Among these constants C the best constant is

$$C(n,2) = \begin{cases} \dfrac{1}{4\pi} \dfrac{\log\alpha_1 - \log\alpha_0}{\alpha_1 - \alpha_0} & (n=2), \\ \dfrac{1}{4\pi} \dfrac{1}{\alpha_0^{1/2} + \alpha_1^{1/2}} & (n=3). \end{cases}$$

If we substitute $C(n,2)$ into C in (3.1)*, the equality holds for $u(x) = K(x,y)$ for every fixed $y \in \mathbf{R}^n$. The reproducing kernel $K(x,y)$ is given by the following formula.*

(3.2)

$$\begin{aligned} K(x,y) &= G(\alpha_0, \alpha_1; x-y) \\ &= \int_0^\infty \frac{1}{\alpha_1 - \alpha_0} \left(e^{-\alpha_0 t} - e^{-\alpha_1 t}\right) H(x-y,t)\, dt. \end{aligned}$$

$$\begin{aligned} &H(x,t) \\ &= \begin{cases} (4\pi t)^{-1} \exp\bigl(-(x_1^2 + x_2^2)/(4t)\bigr) & (n=2) \\ (4\pi t)^{-3/2} \exp\bigl(-(x_1^2 + x_2^2 + x_3^2)/(4t)\bigr) & (n=3). \end{cases} \end{aligned}$$

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