The gradient maps associated to certain non-homogeneous cones

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Abstract: The gradient map associated to a regular open convex cone gives a diffeomorphism from the cone onto its dual cone. If the cone is homogeneous, the inverse of the map is known to be equal to the gradient map associated to the dual cone. However, we show that this is no longer true for a general case by presenting a simple counterexample.

Key words: Regular open convex cone; dual cone; gradient map.

1. Introduction. An open convex cone Ω in a finite-dimensional real vector space V is said to be *regular* if Ω contains no straight line. This is equivalent to saying that the dual cone $\Omega^* := \{\lambda \in V^*\,;\ \lambda(v) > 0 \text{ for all } v \in \overline{\Omega} \setminus \{0\}\}$ is not empty. The characteristic function $\varphi_\Omega : \Omega \to \mathbf{R}$ of a regular cone Ω is defined by an integral over the dual cone: $\varphi_\Omega(v) := \int_{\Omega^*} e^{-\lambda(v)} d\lambda$ ($v \in \Omega$). Vinberg [4, Chapter I, Proposition 7] shows that the gradient map $\iota_\Omega : \Omega \ni v \mapsto -d \log \varphi_\Omega(v) \in V^*$ gives a diffeomorphism from Ω onto Ω^* (see [1] and [3] for generalizations of this map). Since the dual cone of Ω^* coincides with Ω under the identification $(V^*)^* \equiv V$, the gradient map ι_{Ω^*} associated to Ω^* is a diffeomorphism from Ω^* onto Ω. Then it is natural to expect that ι_{Ω^*} equals the inverse map of ι_Ω. Indeed, this is the case if Ω is *homogeneous*, that is, if the group $\mathrm{GL}(\Omega) := \{g \in \mathrm{GL}(V);\ g\Omega = \Omega\}$ acts on Ω transitively ([4, Chapter I, Proposition 10]). For a general regular cone Ω, it has been an open question whether $\iota_{\Omega^*} = \iota_\Omega^{-1}$ ([2, p. 23]). In this article, we shall present a simple counterexample which answers the question in the negative.

2. Preliminaries. For vectors $a_1, a_2, \ldots, a_m \in \mathbf{R}^N$, we denote by $\langle a_1, a_2, \ldots, a_m\rangle_+$ the open convex cone spanned by these vectors:

$$\langle a_1, a_2, \ldots, a_m\rangle_+ := \left\{ \sum_{k=1}^{m} t_k a_k \in \mathbf{R}^N;\ t_1,\ t_2, \ldots, t_m > 0 \right\}.$$

Put

$$v_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad v_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\quad v_3 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\quad v_4 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and consider a cone $\Omega_1 \subset \mathbf{R}^3$ given by

$$\Omega_1 := \langle v_1, v_2, v_3, v_4\rangle_+ . \tag{1}$$

Then it is easy to see that

$$\begin{aligned} \Omega_1 &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3;\ 0 < x < z,\ 0 < y < z \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3;\ \begin{matrix} x > 0, & y > 0 \\ -x + z > 0, & -y + z > 0 \end{matrix} \right\}. \end{aligned} \tag{2}$$

To know directly that Ω_1 is not homogeneous, we note by (1) that the union of the extremal generators ([2, p. 73]) of the closure of Ω_1 equals $\bigcup_{i=1}^{4} \langle v_i \rangle_+$, which is invariant under the action of $\mathrm{GL}(\Omega_1)$. Thus $\mathrm{GL}(\Omega_1)$ preserves also the subset $L := \langle v_1, v_3\rangle_+ \cap \langle v_2, v_4\rangle_+$ of Ω_1. In other words, a point on L cannot be mapped into $\Omega \setminus L$ by any element of $\mathrm{GL}(\Omega_1)$.

Let Ω_2 be the dual cone of Ω_1. We identify the dual space $(\mathbf{R}^3)^*$ with $\mathbf{R}^3$ by the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{R}^3$. Then (1) and (2) tell us that

$$\begin{aligned} \Omega_2 &= \left\{ \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \in \mathbf{R}^3;\ \begin{matrix} \zeta > 0, & \xi + \zeta > 0 \\ \xi + \eta + \zeta > 0, & \eta + \zeta > 0 \end{matrix} \right\} \\ &= \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle_+ , \end{aligned}$$

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where we set

$$\lambda_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \lambda_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \lambda_3 := \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ \lambda_4 := \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Let us calculate the characteristic function φ_1 of Ω_1. For $v \in \Omega_1$, we have by definition $\varphi_1(v) = \int_{\Omega_2} e^{-\langle v,\lambda\rangle}\, d\lambda$. Since $\Omega_2 = \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle_+$ equals the union of the cones $\langle \lambda_1, \lambda_2, \lambda_3 \rangle_+$ and $\langle \lambda_1, \lambda_3, \lambda_4 \rangle_+$ up to a set of measure zero, we have

$$\begin{aligned}&\varphi_1(v)\\ &\quad = \int_{\langle \lambda_1,\lambda_2,\lambda_3\rangle_+} e^{-\langle v,\lambda\rangle}\, d\lambda + \int_{\langle \lambda_1,\lambda_3,\lambda_4\rangle_+} e^{-\langle v,\lambda\rangle}\, d\lambda.\end{aligned}$$

Noting that $\det(\lambda_1\ \lambda_2\ \lambda_3) = 1$, we have

$$\begin{aligned}&\int_{\langle \lambda_1,\lambda_2,\lambda_3\rangle_+} e^{-\langle v,\lambda\rangle} d\lambda\\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-\langle v,\, t_1\lambda_1+t_2\lambda_2+t_3\lambda_3\rangle} dt_1 dt_2 dt_3\\ &= \int_0^{+\infty} e^{-t_1\langle v,\lambda_1\rangle} dt_1 \cdot \int_0^{+\infty} e^{-t_2\langle v,\lambda_2\rangle} dt_2 \cdot\\ &\qquad\qquad \cdot \int_0^{+\infty} e^{-t_3\langle v,\lambda_3\rangle} dt_3\\ &= \frac{1}{\langle v,\lambda_1\rangle\langle v,\lambda_2\rangle\langle v,\lambda_3\rangle}.\end{aligned}$$

Similar calculation leads us to

$$\int_{\langle \lambda_1,\lambda_3,\lambda_4\rangle_+} e^{-\langle v,\lambda\rangle}\, d\lambda = \frac{1}{\langle v,\lambda_1\rangle\langle v,\lambda_3\rangle\langle v,\lambda_4\rangle}.$$

Thus we obtain for ${}^t(x, y, z) \in \Omega_1$

$$\begin{aligned}\varphi_1(x,y,z) &= \frac{1}{xy(z-x)} + \frac{1}{x(z-x)(z-y)}\\ &= \frac{z}{xy(z-x)(z-y)}.\end{aligned} \tag{3}$$

In the same way, we calculate the characteristic function φ_2 of Ω_2. For $\lambda = {}^t(\xi, \eta, \zeta) \in \Omega_2$, we have

$$\varphi_2(\lambda) = \frac{1}{\langle v_1,\lambda\rangle\langle v_2,\lambda\rangle\langle v_3,\lambda\rangle} + \frac{1}{\langle v_1,\lambda\rangle\langle v_3,\lambda\rangle\langle v_4,\lambda\rangle},$$

so that

$$\begin{aligned}&\varphi_2(\xi,\eta,\zeta)\\ &= \frac{1}{\zeta(\xi+\zeta)(\xi+\eta+\zeta)} + \frac{1}{\zeta(\eta+\zeta)(\xi+\eta+\zeta)}\\ &= \frac{\xi+\eta+2\zeta}{\zeta(\xi+\zeta)(\eta+\zeta)(\xi+\eta+\zeta)}.\end{aligned} \tag{4}$$

3. The gradient maps. We denote by ι_1 the gradient map associated to the cone Ω_1. By (3), we have

$$\begin{aligned}&-d\log\varphi_1\\ &= d\{\log x + \log y + \log(z-x) + \log(z-y) - \log z\}\\ &= \Big(\frac{1}{x} - \frac{1}{z-x}\Big)dx + \Big(\frac{1}{y} - \frac{1}{z-y}\Big)dy\\ &\qquad + \Big(\frac{1}{z-x} + \frac{1}{z-y} - \frac{1}{z}\Big)dz,\end{aligned}$$

so that we get

$$\begin{aligned}&\iota_1(x,y,z)\\ &\quad = \Big(\frac{1}{x} - \frac{1}{z-x},\ \frac{1}{y} - \frac{1}{z-y},\ \frac{1}{z-x} + \frac{1}{z-y} - \frac{1}{z}\Big).\end{aligned}$$

Let ι_2 be the gradient map associated to Ω_2. By (4), we have

$$\begin{aligned}&-d\log\varphi_2\\ &= d\{\log\zeta + \log(\xi+\zeta) + \log(\eta+\zeta)\\ &\qquad + \log(\xi+\eta+\zeta) - \log(\xi+\eta+2\zeta)\}\\ &= \Big(\frac{1}{\xi+\zeta} + \frac{1}{\xi+\eta+\zeta} - \frac{1}{\xi+\eta+2\zeta}\Big)d\xi\\ &\quad + \Big(\frac{1}{\eta+\zeta} + \frac{1}{\xi+\eta+\zeta} - \frac{1}{\xi+\eta+2\zeta}\Big)d\eta\\ &\quad + \Big(\frac{1}{\zeta} + \frac{1}{\xi+\zeta} + \frac{1}{\eta+\zeta} + \frac{1}{\xi+\eta+\zeta}\\ &\qquad\qquad - \frac{2}{\xi+\eta+2\zeta}\Big)d\zeta.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\iota_2(\xi,\eta,\zeta)\\ &= \Big(\frac{1}{\xi+\zeta} + \frac{1}{\xi+\eta+\zeta} - \frac{1}{\xi+\eta+2\zeta},\\ &\quad \frac{1}{\eta+\zeta} + \frac{1}{\xi+\eta+\zeta} - \frac{1}{\xi+\eta+2\zeta},\\ &\quad \frac{1}{\zeta} + \frac{1}{\xi+\zeta} + \frac{1}{\eta+\zeta} + \frac{1}{\xi+\eta+\zeta} - \frac{2}{\xi+\eta+2\zeta}\Big).\end{aligned}$$

Now we observe that $\iota_1(1/6, 1/3, 1/2) = (3, -3, 7)$, and that $\iota_2(3, -3, 7) = (6/35, 9/28, 69/140) \neq (1/6, 1/3, 1/2)$. Therefore we obtain

Theorem. *The gradient map ι_2 associated to the dual cone of Ω_1 is not equal to the inverse of the gradient map ι_1 associated to Ω_1.*

This result implies that the gradient map does not necessarily give an isometry between the canonical metrics on a regular cone and its dual cone in the sense of [4, Chapter I, §3].

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