On the structure of Jackson integrals of BC_n type and holonomic q-difference equations

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Abstract: Finiteness of non-symmetric and symmetric cohomologies associated with Jackson integrals of type BC_n is studied. Explicit bases of the cohomologies are also stated. It is shown that the integrals using these bases satisfy holonomic systems of linear q-difference equations with respect to the parameters.

Key words: Jackson integrals of type BC_n; q-de Rham cohomology of type BC_n.

The aim of this note is to explain finite dimensionality and to find bases of non-symmetric and symmetric cohomologies associated with Jackson integrals of type BC_n. More explicitly they are indicated by

$$\dim H^n(X,\Phi,\nabla_q)=\{m+2(n-1)l\}^n,$$
$$\dim H^n_{\mathrm{sym}}(X,\Phi,\nabla_q)=\binom{s+(n-1)l}{n}$$

using terminology in §1.2 of this note. As a consequence, they lead us to the fact that the integrals using these bases satisfy linear holonomic q-difference equations with respect to the parameters. In a generic case, finite dimensionality was proved in full generality in [1, 7, 17]. But here under the condition being a little more restrictive, we show it by constructing a concrete basis (see Theorems 6–9).

Throughout this note, q is a real number such that $0<q<1$ and we use the notation $(a)_i=(a)_\infty/(aq^i)_\infty$, $i\in\mathbf{Z}$ where $(a)_\infty=\prod_{i=0}^\infty(1-aq^i)$. We also use the notations $\tilde\kappa:=\{m+2(n-1)l\}^n$ and $\kappa:=\binom{s+(n-1)l}{n}$.

1. Finiteness of cohomologies of type BC_n.

In order to explain the main theorems we first state the concepts of the Jackson integrals and their cohomologies.

1.1. Jackson integrals.

Let m be an even positive integer $m=2s+2$, $s=-1,0,1,2,3,\dots$ and $a_1,a_2,\dots,a_m$, $t_1,t_2,\dots,t_l$ be arbitrary constants in $\mathbf{C}^*$. We denote by $\Phi(z)=\Phi(z_1,z_2,\dots,z_n)$ the multiplicative function of B_n type

$$\prod_{r=1}^n\left(z_r^{m/2-\delta+(n-r)(l-2\tau)}\prod_{k=1}^m\frac{(qa_k^{-1}z_r;q)_\infty}{(a_kz_r;q)_\infty}\right)$$
$$\times\prod_{k=1}^l\prod_{1\le i<j\le n}\frac{(qt_k^{-1}z_i/z_j;q)_\infty(qt_k^{-1}z_iz_j;q)_\infty}{(t_kz_i/z_j;q)_\infty(t_kz_iz_j;q)_\infty}$$

defined on $X=(\mathbf{C}^*)^n$, where we put

$$q^\delta=a_1a_2\cdots a_m,\quad q^\tau=t_1t_2\cdots t_l.$$

We denote by $\Delta(z)$ the function

$$\prod_{i=1}^n\frac{1-z_i^2}{z_i}\prod_{1\le j<k\le n}\frac{(1-z_j/z_k)(1-z_jz_k)}{z_j},$$

which is called Weyl's denominator of type C_n. For an arbitrary $z=(z_1,z_2,\dots,z_n)\in X$, we define the q-shift $z\to zq^\nu$ by the lattice point $\nu=(\nu_1,\nu_2,\dots,\nu_n)\in\mathbf{Z}^n$ as

$$zq^\nu:=(z_1q^{\nu_1},z_2q^{\nu_2},\dots,z_nq^{\nu_n})\in X.$$

The set $\Lambda_z:=\{zq^\nu\in X\,;\,\nu\in\mathbf{Z}^n\}$ forms an orbit of a lattice subgroup of X.

Definition 1. For a function $\varphi(z)$ on X and an arbitrary point $\xi=(\xi_1,\xi_2,\dots,\xi_n)\in X$, the Jackson integral over the lattice Λ_ξ is defined as the pairing of difference n-forms and lattices

$$\int_{\Lambda_\xi}\Phi(z)\varphi(z)\frac{d_qz_1}{z_1}\wedge\cdots\wedge\frac{d_qz_n}{z_n}:=(1-q)^n\sum_{\nu\in\mathbf{Z}^n}\Phi(\xi q^\nu)\varphi(\xi q^\nu)\tag{1}$$

if it is summable. The LHS of (1) will simply be denoted by $\langle\varphi,\xi\rangle$. Moreover we set

$$\langle\varphi,\xi\rangle_\Delta:=\langle\varphi\Delta,\xi\rangle\tag{2}$$

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where $\varphi\Delta(z) = \varphi(z)\Delta(z)$.

The Weyl group W of type C_n is generated by the reflections

$$\sigma_i : z_i \longleftrightarrow z_{i+1} \quad (1 \le i \le n-1),$$
$$\sigma_n : z_n \longleftrightarrow z_n^{-1}.$$

The group W acts on a space of functions on X by the rule $\sigma f(z) := f(\sigma^{-1}z)$, $\sigma \in W$.

Let $\Theta(z)$ be the functions on X defined by

$$\prod_{r=1}^{n}\left(z_r^{m/2-\delta+(n-r)(l-2\tau)} \prod_{h=1}^{m} \frac{1}{\theta(a_h z_r)} \right) \times \prod_{k=1}^{l} \prod_{1\le i<j\le n} \frac{1}{\theta(t_k z_i/z_j)\theta(t_k z_i z_j)}$$

where $\theta(z) := (z)_\infty (q/z)_\infty$. Since the function $\theta(z)$ has the property $\theta(qz) = -\theta(z)/z$, if we put

$$U_\sigma(z) := \frac{\sigma\Theta(z)}{\Theta(z)} \quad \text{for} \quad \sigma \in W, \tag{3}$$

then $U_\sigma(z)$ are the *cocycle of pseudo-constants*, i.e., constants with respect to the q-shifts $z \to zq^\nu$, $\nu \in \mathbf{Z}^n$. More precisely, by definition of $\Phi(z)$, it follows that the function $\sigma\Phi(z)$ is equal to $\Phi(z)$ up to the pseudo-constant $U_\sigma(z)$ as follows:

$$\sigma\Phi(z) = \Phi(z)U_\sigma(z). \tag{4}$$

In this sense, we regard the function $\Phi(z)$ as *symmetric* with respect to W, and both of $\Phi(z)$ and $\sigma\Phi(z)$ satisfy the same q-difference equations with respect to $z \to zq^\nu$, $\nu \in \mathbf{Z}^n$.

From (1) and (4) and $\sigma\Delta(z) = \operatorname{sgn}(\sigma)\Delta(z)$ we have the following lemma immediately:

Lemma 2. *If $\sigma \in W$, then*

$$\sigma\langle\varphi,\xi\rangle = U_\sigma(\xi)\langle\sigma\varphi,\xi\rangle.$$

In particular, if $\varphi(z)$ is symmetric under the action of W, i.e., $\sigma\varphi(z) = \varphi(z)$, then

$$\sigma\langle\varphi,\xi\rangle_\Delta = \operatorname{sgn}(\sigma)U_\sigma(\xi)\langle\varphi,\xi\rangle_\Delta.$$

1.2. Rational de Rham cohomology of type BC_n. We denote by L the ring of Laurent polynomials $\mathbf{C}[z_1,\ldots,z_n,z_1^{-1},\ldots,z_n^{-1}]$ in z over $\mathbf{C}$. Let R be the L-module generated by the following set of rational functions of z:

$$\bigcup_{h\ge 0}\left\{ \prod_{k=1}^{m}\prod_{j=1}^{n} \frac{(a_k z_j;q)_{-h}}{(qa_k^{-1}z_j;q)_h} \times \prod_{k=1}^{l}\prod_{1\le i<j\le n} \frac{(t_k z_i/z_j;q)_{-h}(t_k z_i z_j;q)_{-h}}{(qt_k^{-1}z_i/z_j;q)_h(qt_k^{-1}z_i z_j;q)_h} \right\}$$

and R_{sym} and R_{alt} be the parts of R consisting of the elements which are symmetric and skew-symmetric under the action of W respectively, i.e.,

$$R_{\mathrm{sym}} := \{\varphi(z) \in R\,;\, \sigma\varphi(z) = \varphi(z), \sigma \in W\},$$
$$R_{\mathrm{alt}} := \{\varphi(z) \in R\,;\, \sigma\varphi(z) = \operatorname{sgn}(\sigma)\varphi(z), \sigma \in W\}.$$

This implies

$$R_{\mathrm{alt}} = R_{\mathrm{sym}}\Delta(z) := \{\varphi(z)\Delta(z)\,;\, \varphi(z) \in R_{\mathrm{sym}}\}.$$

Lemma 3. *For $\varphi(z) \in R$ and $\xi \in X$, the Jackson integral $\langle\varphi,\xi\rangle$ is described as*

$$\langle\varphi,\xi\rangle = f_\varphi(\xi)\Theta(\xi)$$

where $f_\varphi(z)$ is a holomorphic function on X. Moreover, if $\varphi(z) \in R_{\mathrm{sym}}$, then there exists a holomorphic function $g_\varphi(z)$ on X such that

$$\langle\varphi,\xi\rangle_\Delta = g_\varphi(\xi)\Theta_\Delta(\xi)$$

where $\Theta_\Delta(z) := \Theta(z)\theta_\Delta(z)$ and

$$\theta_\Delta(z) := \prod_{r=1}^{n} \frac{\theta(z_r^2)}{z_r} \prod_{1\le i<j\le n} \frac{\theta(z_i/z_j)\theta(z_i z_j)}{z_i}.$$

See [11] for details. Note that the function $\theta_\Delta(z)$ is obviously skew-symmetric, i.e., $\sigma\theta_\Delta(z) = \operatorname{sgn}(\sigma)\theta_\Delta(z)$, so that we have $\sigma\Theta_\Delta(z) = \operatorname{sgn}(\sigma)U_\sigma(z)\Theta_\Delta(z)$.

Let $\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\}$ be the standard basis of $\mathbf{R}^n$. The cocycle function associated with $\Phi(z)$ is defined by $b_\nu(z) := \Phi(zq^\nu)/\Phi(z)$ for $\nu \in \mathbf{Z}^n$, which is the so-called *b-function*. In particular, if $\nu = \varepsilon_r$, $r = 1,2,\ldots,n$, we have

$$b_{\varepsilon_r}(z) = q^{m/2-\delta+(n-r)(l-2\tau)} \prod_{k=1}^{m} \frac{1-a_k z_r}{1-qa_k^{-1}z_r} \times \prod_{k=1}^{l}\left(\prod_{j=1}^{r-1} \frac{(1-t_k^{-1}z_j/z_r)(1-t_k z_j z_r)}{(1-q^{-1}t_k z_j/z_r)(1-qt_k^{-1}z_j z_r)} \times \prod_{j=r+1}^{n} \frac{(1-t_k z_r/z_j)(1-t_k z_j z_r)}{(1-qt_k^{-1}z_r/z_j)(1-qt_k^{-1}z_j z_r)} \right),$$

which will simply be denoted by $b_r(z)$.

Let $\nabla_q : R^n \to R$ be the n-dimensional covariant q-differenciation defined by

$$\nabla_q\colon (\psi_1(z),\psi_2(z),\dots,\psi_n(z))\mapsto \sum_{j=1}^n \nabla_{q,j}\psi_j(z)$$

where $\nabla_{q,j}\psi(z):=\psi(z)-b_j(z)T_{z_j}\psi(z)$. We denote by $\mathcal{A}\colon R\to R_{\rm alt}$ the alternation

$$\mathcal{A}\colon f(z)\mapsto \sum_{\sigma\in W}\mathrm{sgn}(\sigma)\sigma f(z)$$

for a function $f(z)$ on X. Then we have

$$R_{\rm alt}=\mathcal{A}R,$$
$$\mathcal{A}\nabla_q(R^n)=\nabla_q(R^n)\cap R_{\rm alt}.$$

Definition 4. The quotient $H=R/\nabla_q(R^n)$ and $H_{\rm sym}=R_{\rm alt}/\mathcal{A}\nabla_q(R^n)$ define the n-dimensional non-symmetric and symmetric rational de Rham cohomologies $H^n(X,\Phi,\nabla_q)$ and $H^n_{\rm sym}(X,\Phi,\nabla_q)$ associated with the Jackson integrals (1) respectively, because they are isomorphic to each other (see also [3, 7] for the definitions of these cohomologies).

Remark 4.1. Because of symmetry, it follows that

$$\mathcal{A}\nabla_q(R^n)\subset\nabla_q(R^n)$$

and that all $\mathcal{A}\nabla_{q,r}$ are the same for $r=1,2,\dots,n$, so that we have

$$\mathcal{A}\nabla_q(R^n)=\mathcal{A}\nabla_{q,r}R.$$

This implies that $H_{\rm sym}$ is identified with the linear subspace of H consisting of the elements which are skew-symmetric under the Weyl group W.

Lemma 5. *Suppose $\varphi(z)\in\nabla_q(R^n)$. Then*

$$\langle\varphi,\xi\rangle=0\quad\textit{and}\quad\langle\mathcal{A}\varphi,\xi\rangle=0$$

if it is summable.

This lemma shows that the integral $\langle\varphi,\xi\rangle$ for $\varphi(z)\in R$ and that for $\varphi(z)\in R_{\rm alt}$ depend only on the quotients H and $H_{\rm sym}$ respectively.

1.3. Regularization of Jackson integrals. We denote by $\mathcal{H}$ the linear space of holomorphic functions $f(z)$ on X satisfying

$$T_{z_i}f(z)=\big(qz_i^2\big)^{-m/2-(n-1)l}f(z)$$

for $i=1,2,\dots,n$. The space $\mathcal{H}$ has the dimension $\tilde\kappa$. Let $\mathcal{H}_{\rm sym}$ be the linear space of holomorphic functions $f(z)$ on X satisfying $\sigma f(z)=f(z)$ and

$$T_{z_i}f(z)=\big(qz_i^2\big)^{-m/2-(n-1)l+n+1}f(z)$$

for $i=1,2,\dots,n$. The space $\mathcal{H}_{\rm sym}$ has the dimension κ. By definition, the Jackson integrals $\langle\varphi,z\rangle$ and $\langle\varphi,z\rangle_\Delta$ are meromorhic as functions on X. For $\langle\varphi,z\rangle$ and $\langle\varphi,z\rangle_\Delta$ we define the *regularized Jackson integrals* as follows respectively:

$$\langle\!\langle\varphi,z\rangle\!\rangle:=\langle\varphi,z\rangle/\Theta(z),$$
$$\langle\!\langle\varphi,z\rangle\!\rangle_\Delta:=\langle\varphi,z\rangle_\Delta/\Theta_\Delta(z).$$

Lemma 3 implies that $\langle\!\langle\varphi,z\rangle\!\rangle\in\mathcal{H}$ and that $\langle\!\langle\varphi,z\rangle\!\rangle_\Delta\in\mathcal{H}_{\rm sym}$ if $\varphi\in R_{\rm sym}$, so that they are holomorphic functions on X.

1.4. Symplectic Schur functions. For a sequence of integers $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbf{Z}^n$ we set $z^\lambda:=z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n}$. Let Q be the set defined by

$$\left\{\lambda\in\mathbf{Z}^n\,;\ \begin{matrix}-s-1-(n-1)l\le\lambda_i\le s+(n-1)l\\ \text{for}\quad i=1,2,\dots,n\end{matrix}\right\},$$

which consists of $\tilde\kappa$ elements. We denote the skew-symmetric Laurent polynomials in z

$$\mathcal{A}z^\lambda:=\sum_{\sigma\in W}\mathrm{sgn}(\sigma)\sigma(z^\lambda).$$

The Weyl denominator formula says that

$$\mathcal{A}z^\rho=(-1)^n\Delta(z)$$

where $\rho=(n,n-1,\dots,2,1)\in\mathbf{Z}^n$. Let P be the set of all partitions defined by $\{\lambda\in\mathbf{Z}^n\,;\,s-1+(n-1)(l-1)\ge\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n\ge0\}$, which consists of κ elements. We define the *symplectic Schur function*

$$\chi_\lambda(z):=\frac{\mathcal{A}z^{\lambda+\rho}}{\mathcal{A}z^\rho}$$

which occurs in the Weyl character formula.

1.5. Main results. In the sequel we assume that

($\mathcal{C}$) *all the parameters $a_1,a_2,\dots,a_m$ and $t_1,t_2,\dots,t_l$ are generic.*

The following four theorems are the main results of this note:

Theorem 6. *Under the condition ($\mathcal{C}$), $H^n(X,\Phi,\nabla_q)$ has dimension $\tilde\kappa=\{m+2(n-1)l\}^n$ and is spanned by the basis $\{z^\lambda\,;\,\lambda\in Q\}$.*

Theorem 7. *Under the condition ($\mathcal{C}$), $H^n_{\rm sym}(X,\Phi,\nabla_q)$ has dimension $\kappa=\binom{s+(n-1)l}{n}$ and is spanned by the basis $\{\chi_\lambda(z)\Delta(z)\,;\,\lambda\in P\}$.*

We denote by T_u the shift operator on a parameter $u\to uq$. From Theorems 6 and 7 we have the holonomic q-difference equations for $\langle z^\lambda,\xi\rangle$ and $\langle\chi_\lambda,\xi\rangle_\Delta$ respectively, with respect to the q-shift of the parameters $a_1,a_2,\dots,a_m,t_1,t_2,\dots,t_l$ as follows:

Theorem 8. *There exist invertible matrices $\mathcal{Y}_{a_k}$, $\mathcal{Y}_{t_j}$ whose components $\eta^{(a_k)}_{\lambda,\nu},\eta^{(t_j)}_{\lambda,\nu}$ are rational* *functions of* $a_1,\dots,a_m,t_1,\dots,t_l$ *respectively, such that*

$$T_{a_k}\langle z^\lambda,\xi\rangle=\sum_{\nu\in Q}\eta^{(a_k)}_{\lambda,\nu}\langle z^\nu,\xi\rangle,$$

$$T_{t_j}\langle z^\lambda,\xi\rangle=\sum_{\nu\in Q}\eta^{(t_j)}_{\lambda,\nu}\langle z^\nu,\xi\rangle$$

where λ *runs over the set* Q.

Theorem 9. *There exist invertible matrices* Y_{a_k},Y_{t_j} *whose components* $y^{(a_k)}_{\lambda,\nu},y^{(t_j)}_{\lambda,\nu}$ *are rational functions of* $a_1,\dots,a_m,t_1,\dots,t_l$ *respectively, such that*

$$T_{a_k}\langle\chi_\lambda,\xi\rangle_\Delta=\sum_{\nu\in P}y^{(a_k)}_{\lambda,\nu}\langle\chi_\nu,\xi\rangle_\Delta,\tag{5}$$

$$T_{t_j}\langle\chi_\lambda,\xi\rangle_\Delta=\sum_{\nu\in P}y^{(t_j)}_{\lambda,\nu}\langle\chi_\nu,\xi\rangle_\Delta\tag{6}$$

where λ *runs over the set* P.

Remark 9.1. When $(m,l)=(2n+2,0)$ or $(4,1)$ in Theorem 7 the number κ equals 1 and hence the matrices Y_{a_k} and Y_{t_1} in Theorem 9 reduce to scalars which are explicitly expressible as ratios of products of q-gamma functions. These coincide with some of the results in [8–10, 12–15, etc.]. See also Theorems 10 and 11 in the next section.

The proofs of Theorems 6–9 are given in [5] by indicating the isomorphisms

$$H\xrightarrow{\sim}\mathcal{H},\quad H_{\rm sym}\xrightarrow{\sim}\mathcal{H}_{\rm sym},$$

which are based on the results in [2, 7].

2. Special symmetric cases. We consider the map

$$\begin{aligned}\mathcal{M}_{\rm sym}:\ R_{\rm sym}\Delta(z)&\to\ \mathcal{H}_{\rm sym}\\ \varphi(z)\Delta(z)&\mapsto\langle\!\langle\varphi,z\rangle\!\rangle_\Delta,\end{aligned}$$

which is well-defined from Eq.(3), Lemmas 2 and 3. Since we see in [5] that $\operatorname{Ker}\mathcal{M}_{\rm sym}=\mathcal{A}\nabla_q(R^n)$, the map $\mathcal{M}_{\rm sym}$ naturally induces the isomorphism $H_{\rm sym}\xrightarrow{\sim}\mathcal{H}_{\rm sym}$.

Using the map $\mathcal{M}_{\rm sym}$, Eqs.(5) and (6) in Theorem 9 are rewritten as the equations in $\mathcal{H}_{\rm sym}$ as follows:

$$T_{a_k}\langle\!\langle\chi_\lambda,\xi\rangle\!\rangle_\Delta=\sum_{\nu\in P}\bar{y}^{(a_k)}_{\lambda,\nu}\langle\!\langle\chi_\nu,\xi\rangle\!\rangle_\Delta,$$

$$T_{t_j}\langle\!\langle\chi_\lambda,\xi\rangle\!\rangle_\Delta=\sum_{\nu\in P}\bar{y}^{(t_j)}_{\lambda,\nu}\langle\!\langle\chi_\nu,\xi\rangle\!\rangle_\Delta,$$

and $\overline{Y}_{a_k}:=\left(\bar{y}^{(a_k)}_{\lambda,\nu}\right),\overline{Y}_{t_j}:=\left(\bar{y}^{(t_j)}_{\lambda,\nu}\right)$ denote square matrices of degree $\kappa=\binom{s+(n-1)l}{n}$ whose components are rational functions of $a_1,a_2,\dots,a_m,t_1,t_2,\dots,t_l$ respectively.

The following two facts are essential for proving the isomorphism $H_{\rm sym}\xrightarrow{\sim}\mathcal{H}_{\rm sym}$ in [5]. One is that $\overline{Y}_{a_k}$, $\overline{Y}_{t_j}$ are invertible, i.e., $\det\overline{Y}_{a_k}$, $\det\overline{Y}_{t_j}$ do not vanish identically. The other is that the map $\mathcal{M}_{\rm sym}$ does not degenerate, i.e., the functions $\langle\!\langle\chi_\lambda,z\rangle\!\rangle_\Delta$, $\lambda\in P$ are linearly independent in $\mathcal{H}_{\rm sym}$. This is equivalent to the fact $\det\left(\langle\!\langle\chi_\lambda,\zeta_{(\mu)}\rangle\!\rangle_\Delta\right)_{\lambda,\mu}$ does not vanish identically for some κ points $\zeta_{(\mu)}$ in X.

In this section, we mention more concrete results about them when $l=0$ and 1.

2.1. Symmetric case where $l=0$. In this case, $H^n_{\rm sym}(X,\Phi,\nabla_q)$ has dimension $\kappa=\binom{s}{n}$. According to the following theorem, we see directly that $\det\overline{Y}_{a_k}$ and $\det\left(\langle\!\langle\chi_\lambda,\zeta_{(\mu)}\rangle\!\rangle_\Delta\right)_{\lambda,\mu}$ do not vanish identically:

Theorem 10. *The explicit form of* $\det\overline{Y}_{a_k}$ *is given by*

$$\det\overline{Y}_{a_k}=\left(\frac{\prod_{i=1}^{2s+2}\left(1-a_k^{-1}a_i^{-1}\right)}{\left(1-a_k^{-2}\right)\left(1-a_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1}\right)}\right)^{\binom{s-1}{n-1}}.$$

Moreover, the $\kappa\times\kappa$ *determinant with* (λ,μ) *entry* $\langle\!\langle\chi_\lambda,\zeta_{(\mu)}\rangle\!\rangle_\Delta$ *is evaluated as*

$$\begin{aligned}&\{(1-q)(q)_\infty\}^{n\binom{s}{n}}\\ &\times\left(\frac{\prod_{1\le i<j\le 2s+2}\left(qa_i^{-1}a_j^{-1}\right)_\infty}{\left(qa_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1}\right)_\infty}\right)^{\binom{s-1}{n-1}}\\ &\times\left(\prod_{1\le i<j\le s}\frac{\theta(a_i/a_j)\theta(a_ia_j)}{a_i}\right)^{\binom{s-2}{n-1}}\end{aligned}$$

where $\zeta_{(\mu)}:=(a_{\mu_1+n},a_{\mu_2+n-1},\dots,a_{\mu_n+1})\in X$ *for* $\mu=(\mu_1,\mu_2,\dots,\mu_n)\in P$.

Proof. See [4]. $\square$

Remark 10.1. When $m=2n+2$, i.e., $s=n$, the above determinant, whose matrix size $\binom{s}{n}$ equals 1, becomes nothing but the formula investigated by Gustafson [10]. See also [15].

2.2. Symmetric case where $l=1$. We shall simply write t in place of t_1. In this case, $H^n_{\rm sym}(X,\Phi,\nabla_q)$ has dimension $\kappa=\binom{s+n-1}{n}$. The following implies that $\det\overline{Y}_{a_k}$ does not vanish identically:

Theorem 11. *The explicit form of* $\det\overline{Y}_{a_k}$ *is given by*

$$\prod_{j=1}^{n}\left(\frac{\prod_{i=1}^{2s+2}\left(1-t^{j-n}a_k^{-1}a_i^{-1}\right)}{\left(1-t^{j-n}a_k^{-2}\right)\left(1-t^{2-n-j}a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1}\right)}\right)^{\binom{s+j-2}{j-1}}.$$

Proof. See [6]. $\square$

Next we show the explicit form of the determinant $\det\left(\langle\!\langle \chi_\lambda, \zeta_{(\mu)}\rangle\!\rangle_\Delta\right)_{\lambda,\mu}$ for some κ points $\zeta_{(\mu)}$ in X. In order to explain it, we choose special critical points $\zeta_{(\mu)}$ for the Jackson integrals (2) in the following manner.

Let Z be the set of all s-tuples defined by

$$\left\{(\mu_1,\mu_2,\ldots,\mu_s)\in \mathbf{Z}^s\,;\ \begin{array}{l}\mu_1+\cdots+\mu_s=n,\\ \mu_1\ge 0,\ldots,\mu_s\ge 0\end{array}\right\},$$

which consists of κ elements. For s-tuples $\mu = (\mu_1,\mu_2,\ldots,\mu_s)$ and $\nu=(\nu_1,\nu_2,\ldots,\nu_s)\in Z$, we define the ordering $\mu \prec_Z \nu$ on Z if there exists i such that $\mu_1=\nu_1,\mu_2=\nu_2,\ldots,\mu_{i-1}=\nu_{i-1}$, $\mu_i<\nu_i$. Corresponding to the s-tuple $\mu=(\mu_1,\mu_2,\ldots,\mu_s)\in Z$, we take the point $(\zeta_1,\zeta_2,\ldots,\zeta_n)\in X$ satisfying

$$\zeta_i=\begin{cases} a_1t^{\mu_1-i} & \text{if } 1\le i\le \mu_1,\\ a_2t^{\mu_1+\mu_2-i} & \text{if } \mu_1+1\le i\le \mu_1+\mu_2,\\ \quad\vdots & \qquad\vdots\\ a_st^{n-i} & \text{if } \displaystyle\sum_{k=1}^{s-1}\mu_k+1\le i\le n.\end{cases}$$

We denote by $\zeta_{(\mu)}=(\zeta_{(\mu)1},\zeta_{(\mu)2},\ldots,\zeta_{(\mu)n})\in X$ such a point.

For $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\nu=(\nu_1,\nu_2,\ldots,\nu_n)\in P$, we also define the reverse lexicographic ordering $\lambda\prec\nu$ on P if $\lambda_1=\nu_1$, $\lambda_2=\nu_2,\ldots,\lambda_{i-1}=\nu_{i-1}$, $\lambda_i<\nu_i$ for some $i\in\{1,2,\ldots,n\}$.

Theorem 12. *The $\kappa\times\kappa$ determinant with (λ,μ) entry $\langle\!\langle \chi_\lambda,\zeta_{(\mu)}\rangle\!\rangle_\Delta$ is evaluated as*

$$\begin{aligned}&\{(1-q)(q)_\infty\}^{n\binom{s+n-1}{n}}\\ &\times\prod_{k=1}^{n}\left(\frac{\left(qt^{-(n-k+1)}\right)_\infty^s}{\left(qt^{-1}\right)_\infty^s}\right.\\ &\qquad\left.\times\frac{\prod_{1\le i<j\le 2s+2}\left(qt^{-(n-k)}a_i^{-1}a_j^{-1}\right)_\infty}{\left(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\cdots a_{2s+2}^{-1}\right)_\infty}\right)^{\binom{s+k-2}{k-1}}\\ &\quad\times\prod_{k=1}^{n}\left(\prod_{r=0}^{n-k}\prod_{1\le i<j\le s}\frac{\theta\left(t^{2r-(n-k)}a_ia_j^{-1}\right)}{t^ra_i}\right.\\ &\qquad\qquad\left.\times\,\theta\left(t^{n-k}a_ia_j\right)\right)^{\binom{s+k-3}{k-1}},\end{aligned}$$

where the rows $\lambda\in P$ and the columns $\mu\in Z$ of the matrix $\det\left(\langle\!\langle \chi_\lambda,\zeta_{(\mu)}\rangle\!\rangle_\Delta\right)_{\lambda,\mu}$ *are arranged in the decreasing orders of $\prec$ and $\prec_Z$ respectively.*

Proof. See [6]. $\square$

As a corollary, we see $\det\left(\langle\!\langle \chi_\lambda,\zeta_{(\mu)}\rangle\!\rangle_\Delta\right)_{\lambda,\mu}$ does not vanish identically.

Remark 12.1. In the special case where $(m,l)=(4,1)$, i.e., $(s,l)=(1,1)$, κ is equal to 1, and the determinant reduces to Jackson integral itself which is explicitly evaluated by van Diejen [9]. See also [8, 13, 14, etc.].

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