A note on isometric immersions of the Cayley projective plane and Frenet curves

By Hiromasa TANABE

Department of Mathematics, Shimane University
1060, Nishikawatsu-cho, Matsue, Shimane 690-8504
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Abstract: We give a characterization of the first standard imbedding of the Cayley projective plane into a real space form in terms of a particular class of Frenet curves of order 2.

Key words: Cayley projective plane; parallel isometric immersions; Frenet curves.

1. Introduction. Let $f : M \to \widetilde{M}$ be an isometric immersion of a Riemannian manifold M into an ambient Riemannian manifold $\widetilde{M}$. By observing the extrinsic shape of curves of a submanifold M, we can study the properties of the immersion $f : M \to \widetilde{M}$ in some cases.

In their paper [5], Nomizu and Yano proved a well-known theorem which states that a submanifold M is an extrinsic sphere of $\widetilde{M}$, namely M is a totally umbilic submanifold with parallel mean curvature vector in $\widetilde{M}$, if and only if all circles of some constant positive curvature k in M are circles in the ambient space $\widetilde{M}$. In [3], Kôzaki and Maeda improved this theorem, that is, they showed that M is an extrinsic sphere of $\widetilde{M}$ if and only if all circles of some constant positive curvature k in M are Frenet curves of proper order 2 in $\widetilde{M}$.

Then, if an isometric immersion $f : M \to \widetilde{M}$ maps some Frenet curves of proper order 2 on M to Frenet curves of proper order 2 in the ambient space $\widetilde{M}$, what can we say about the immersion f? In the preceding paper [9], the author showed that M is a totally geodesic submanifold of $\widetilde{M}$ if and only if all Frenet curves of proper order 2 of some nonconstant positive curvature function κ in M are mapped to Frenet curves of proper order 2 in the ambient space $\widetilde{M}$. In [4] and [10], S. Maeda and the author characterized parallel isometric immersions of complex projective spaces and quaternionic projective spaces into a real space form by using particular classes of Frenet curves of order 2 which are closely related to the complex structure and the quaternionic Kähler structure of M.

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The purpose of this note is to give a characterization of the first standard imbedding of the Cayley projective plane $\mathbf{C}ayP^2(c)$ into a real space form along this context, which is an improvement of a result of T. Adachi, S. Maeda and K. Ogiue [1].

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2. Cayley circles in the Cayley projective plane. We first review the definition of Frenet curves. A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist a field of orthonormal frames $\{V_1 = \dot\gamma, V_2, \dots, V_d\}$ along γ and positive smooth functions $\kappa_1(s), \dots, \kappa_{d-1}(s)$ satisfying the following system of ordinary differential equations

$$\nabla_{\dot\gamma} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \dots, d,$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The functions $\kappa_j(s)$ $(j = 1, \dots, d-1)$ and a field of orthonormal frames $\{V_1, \dots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. We note that for a given orthonormal frame $\{v_1, \dots, v_d\}$ at any point x of M and positive smooth functions $\kappa_1(s), \dots, \kappa_{d-1}(s)$ there exists a unique Frenet curve $\gamma = \gamma(s)$ defined for some open interval $(-\varepsilon, \varepsilon)$ such that $\gamma(0) = x$, its curvatures are $\kappa_1(s), \dots, \kappa_{d-1}(s)$ and its Frenet frame is coincident with $\{v_1, \dots, v_d\}$ at x. A Frenet curve is called a Frenet curve *of order d* if it is a Frenet curve of proper order r $(\leq d)$. We call a curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic. A helix of order 2, that is a curve which satisfies $\nabla_{\dot\gamma}\dot\gamma = kV_2(s)$, $\nabla_{\dot\gamma}V_2(s) = -k\dot\gamma$, is called a *circle* of curvature k.

A curve $\gamma = \gamma(s)$ on a Riemannian manifold M is called a *plane curve* if the curve γ is locally contained in some 2-dimensional totally geodesic submanifold of M. As a matter of course, every plane curve with positive curvature function is a Frenet curve of proper order 2. But in general, the converse does not hold. In the case that the space M is a real space form $\widetilde{M}^n(\tilde c)$ of constant curvature $\tilde c$ (that is, $\widetilde{M}^n(\tilde c)$ is locally congruent to a Euclidean space $\mathbf{R}^n$, a standard sphere $S^n(\tilde c)$, or a real hyperbolic space $H^n(\tilde c)$ according as the curvature $\tilde c$ is zero, positive, or negative), it is easy to see that a curve γ is a Frenet curve of proper order 2 if and only if the curve γ is a plane curve with positive curvature function.

In the following, we consider a Frenet curve $\gamma = \gamma(s)$ of proper order 2 in the Cayley projective plane $\mathbf{C}ayP^2(c)$ of maximal sectional curvature c which satisfies

$$\nabla_{\dot\gamma}\dot\gamma = \kappa(s)V \quad \text{and} \quad \nabla_{\dot\gamma}V = -\kappa(s)\dot\gamma \tag{2.1}$$

with a unit vector field $V = V(s)$ along γ and a function $\kappa = \kappa(s)$ (> 0). We can see from (2.1) that the sectional curvature $K(\dot\gamma, V)$ given by the osculating plane spanned by $\dot\gamma$ and V is constant along γ. Indeed, since $\nabla R \equiv 0$ we have

$$\begin{aligned}
&\nabla_{\dot\gamma}\langle R(\dot\gamma, V)V, \dot\gamma\rangle \\
&= \langle R(\nabla_{\dot\gamma}\dot\gamma, V)V + R(\dot\gamma, \nabla_{\dot\gamma}V)V + R(\dot\gamma, V)\nabla_{\dot\gamma}V,\ \dot\gamma\rangle \\
&\quad + \langle R(\dot\gamma, V)V, \nabla_{\dot\gamma}\dot\gamma\rangle \\
&= \kappa\langle R(V, V)V, \dot\gamma\rangle - \kappa\langle R(\dot\gamma, \dot\gamma)V, \dot\gamma\rangle \\
&\quad - \kappa\langle R(\dot\gamma, V)\dot\gamma, \dot\gamma\rangle + \kappa\langle R(\dot\gamma, V)V, V\rangle \\
&= 0.
\end{aligned}$$

We here consider the case that the curvature κ of a Frenet curve γ of proper order 2 is constant, namely we study a circle. The circle γ which satisfies $K(\dot\gamma, V) = c$ is called a *Cayley circle*. Suppose that an open subset M of $\mathbf{C}ayP^2(c)$ is isometrically immersed into a real space form $\widetilde{M}^{16+p}(\tilde c)$. It is needless to say that the image of a circle on M is not always a circle in the ambient space $\widetilde{M}^{16+p}(\tilde c)$. However, if the isometric immersion is given by a composition of the first standard minimal imbedding and a totally umbilic imbedding, the image of a Cayley circle on M is a circle in $\widetilde{M}^{16+p}(\tilde c)$ ([1]).

Finally we summarize a few fundamental notions in submanifold theory. Let M and $\widetilde{M}$ be Riemannian manifolds and $f : M \to \widetilde{M}$ an isometric immersion. The Riemannian metrics on M, $\widetilde{M}$ are denoted by the same notation $\langle\ ,\ \rangle$. We denote by ∇ and $\widetilde\nabla$ the covariant differentiations of M and $\widetilde{M}$, respectively. Then the formulas of Gauss and Weingarten are

$$\widetilde\nabla_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde\nabla_X \xi = -A_\xi X + D_X\xi,$$

where σ, A_ξ and D denote the second fundamental form of f, the shape operator in the direction of ξ and the covariant differentiation in the normal bundle, respectively. We define the covariant differentiation $\bar\nabla$ of the second fundamental form σ with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) as follows:

$$\begin{aligned}
(\bar\nabla_X\sigma)(Y, Z) = D_X(\sigma(Y, Z))& \\
- \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)&.
\end{aligned}$$

If $\bar\nabla\sigma = 0$, an isometric immersion f is called *parallel*.

An isometric immersion f is said to be *isotropic* at $x \in M$ if $\|\sigma(v, v)\|/\|v\|^2$ does not depend on the choice of $v\,(\neq 0) \in T_xM$. In this case we put the number as $\lambda(x)$. If the immersion is isotropic at every point, then the immersion is said to be isotropic. When the function $\lambda = \lambda(x)$ is constant on M, we call M a λ-*isotropic* submanifold. Note that a totally umbilic immersion is isotropic, but not *vice versa* ([6]).

3. Main result. We shall prove the following theorem.

Theorem. *Let M be an open subset of the Cayley projective plane $\mathbf{C}ayP^2(c)$ and f an isometric immersion of M into a real space form $\widetilde{M}^{16+p}(\tilde c)$. Suppose that every Frenet curve $\gamma = \gamma(s)$ of proper order 2 on M with some curvature function $\kappa = \kappa(s) > 0$ which satisfies $K(\dot\gamma, V) = c$ is mapped to a plane curve in $\widetilde{M}^{16+p}(\tilde c)$. Then the curve γ is a Cayley circle and the immersion f is locally congruent to a parallel immersion defined by*

(3.1)

$$f_2 \circ f_1 : \mathbf{C}ayP^2(c) \xrightarrow{f_1} S^{25}\left(\frac{3c}{4}\right) \xrightarrow{f_2} \widetilde{M}^{16+p}(\tilde c),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion and $3c/4 \geq \tilde c$.

The basic idea of proof is similar to that of Theorem 2 in [4]. But for readers we explain it in detail.

Proof. Let x be an arbitrary point of M and $v \in T_xM$ an arbitrary unit vector. Let $\gamma = \gamma(s)$ be a Frenet curve of proper order 2 on M satisfying the equations (2.1) and the initial condition $\gamma(0) = x$, $\dot\gamma(0) = v$ and $K(v, V(0)) = c$. Since the curve $f \circ \gamma$ is a plane curve in $\widetilde{M}^{16+p}(\tilde c)$ by assumption, there exist a (nonnegative) function $\tilde\kappa = \tilde\kappa(s)$ and a field of unit vectors $\widetilde V = \widetilde V(s)$ along $f \circ \gamma$ in $\widetilde{M}^{16+p}(\tilde c)$ which satisfy that

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \tilde\kappa\widetilde V, \quad \widetilde\nabla_{\dot\gamma}\widetilde V = -\tilde\kappa\dot\gamma. \tag{3.2}$$

Then using the formula of Gauss, we have

$$\tilde\kappa\widetilde V = \kappa V + \sigma(\dot\gamma, \dot\gamma), \tag{3.3}$$

hence

$$\tilde\kappa^2 = \kappa^2 + \|\sigma(\dot\gamma, \dot\gamma)\|^2. \tag{3.4}$$

The function $\tilde\kappa$ is positive because $\kappa > 0$.

For the left-hand side of (3.3), by using (3.2) and (3.3) again, we get

$$\begin{aligned}\tilde\kappa\widetilde\nabla_{\dot\gamma}(\tilde\kappa\widetilde V) &= \tilde\kappa\{\dot{\tilde\kappa}\widetilde V + \tilde\kappa\widetilde\nabla_{\dot\gamma}\widetilde V\} = \tilde\kappa\dot{\tilde\kappa}\widetilde V - \tilde\kappa^3\dot\gamma \\ &= \dot{\tilde\kappa}\{\kappa V + \sigma(\dot\gamma, \dot\gamma)\} - \tilde\kappa^3\dot\gamma.\end{aligned} \tag{3.5}$$

And for the right-hand side of (3.3), by the formulas of Gauss and Weingarten we have

$$\begin{aligned}&\tilde\kappa\widetilde\nabla_{\dot\gamma}\{\kappa V + \sigma(\dot\gamma, \dot\gamma)\} \\ &= \tilde\kappa\big\{\dot\kappa V + \kappa\widetilde\nabla_{\dot\gamma}V - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma, \dot\gamma))\big\} \\ &= \tilde\kappa\big\{\dot\kappa V + \kappa\,(\nabla_{\dot\gamma}V + \sigma(\dot\gamma, V)) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma \\ &\qquad + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma)\big\} \\ &= \tilde\kappa\big\{\dot\kappa V - \kappa^2\dot\gamma + 3\kappa\sigma(\dot\gamma, V) \\ &\qquad - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma)\big\}.\end{aligned} \tag{3.6}$$

By comparing the tangential components and the normal components for the submanifold M in (3.5) and (3.6), we obtain the following equations:

$$\dot{\tilde\kappa}\kappa V - \tilde\kappa^3\dot\gamma = \tilde\kappa\big\{\dot\kappa V - \kappa^2\dot\gamma - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma\big\}, \tag{3.7}$$

$$\dot{\tilde\kappa}\sigma(\dot\gamma, \dot\gamma) = \tilde\kappa\big\{3\kappa\sigma(\dot\gamma, V) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma)\big\}. \tag{3.8}$$

From (3.4) we have

$$\begin{aligned}\tilde\kappa\dot{\tilde\kappa} &= \frac{1}{2}\frac{d}{ds}\tilde\kappa^2 \\ &= \kappa\dot\kappa + \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &= \kappa\dot\kappa + \langle D_{\dot\gamma}(\sigma(\dot\gamma, \dot\gamma)), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &= \kappa\dot\kappa + \langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &\qquad + 2\kappa\langle\sigma(V, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle.\end{aligned} \tag{3.9}$$

On the other hand, the equation (3.8) yields

$$\tilde\kappa\dot{\tilde\kappa}\sigma(\dot\gamma, \dot\gamma) = \tilde\kappa^2\big\{3\kappa\sigma(\dot\gamma, V) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma)\big\}. \tag{3.10}$$

Substitute (3.4) and (3.9) into (3.10) and set $s = 0$. Then we have

$$\begin{aligned}&\big\{\kappa(0)\dot\kappa(0) + \langle(\bar\nabla_v\sigma)(v, v), \sigma(v, v)\rangle \\ &\quad + 2\kappa(0)\langle\sigma(v, v), \sigma(v, V(0))\rangle\big\}\sigma(v, v) \\ &= \big\{\kappa(0)^2 + \|\sigma(v, v)\|^2\big\}\times \\ &\quad \big\{3\kappa(0)\sigma(v, V(0)) + (\bar\nabla_v\sigma)(v, v)\big\}.\end{aligned} \tag{3.11}$$

Now we note that there exists another Frenet curve $\gamma_1 = \gamma_1(s)$ of proper order 2 with the same curvature κ on M satisfying $\nabla_{\dot\gamma_1}\dot\gamma_1 = \kappa V_1$ and $\nabla_{\dot\gamma_1}V_1 = -\kappa\dot\gamma_1$ with the initial condition $\gamma_1(0) = x$, $\dot\gamma_1(0) = v$ and $V_1(0) = -V(0)$. Then the equality (3.11) for γ_1 turns to

$$\begin{aligned}&\big\{\kappa(0)\dot\kappa(0) + \langle(\bar\nabla_v\sigma)(v, v), \sigma(v, v)\rangle \\ &\quad - 2\kappa(0)\langle\sigma(v, v), \sigma(v, V(0))\rangle\big\}\sigma(v, v) \\ &= \big\{\kappa(0)^2 + \|\sigma(v, v)\|^2\big\}\times \\ &\quad \big\{-3\kappa(0)\sigma(v, V(0)) + (\bar\nabla_v\sigma)(v, v)\big\}.\end{aligned} \tag{3.11$'$}$$

Therefore, from (3.11) and (3.11′) we obtain

$$\begin{aligned}&2\kappa(0)\langle\sigma(v, v), \sigma(v, V(0))\rangle\sigma(v, v) \\ &\quad = 3\kappa(0)\big\{\kappa(0)^2 + \|\sigma(v, v)\|^2\big\}\sigma(v, V(0)),\end{aligned}$$

so that

$$\begin{aligned}&2\langle\sigma(v, v), \sigma(v, V(0))\rangle\sigma(v, v) \\ &\quad = 3\big\{\kappa(0)^2 + \|\sigma(v, v)\|^2\big\}\sigma(v, V(0)).\end{aligned} \tag{3.12}$$

Taking the inner product of both sides of (3.12) with $\sigma(v, v)$, we get

$$\begin{aligned}&2\langle\sigma(v, v), \sigma(v, V(0))\rangle\|\sigma(v, v)\|^2 \\ &\quad = 3\big\{\kappa(0)^2 + \|\sigma(v, v)\|^2\big\}\langle\sigma(v, v), \sigma(v, V(0))\rangle\end{aligned}$$

and hence

$$\big\{3\kappa(0)^2 + \|\sigma(v, v)\|^2\big\}\langle\sigma(v, v), \sigma(v, V(0))\rangle = 0.$$

Since $3\kappa(0)^2 + \|\sigma(v, v)\|^2 > 0$, we have $\langle\sigma(v, v), \sigma(v, V(0))\rangle = 0$. Thus, again from (3.12), we see that $\sigma(v, V(0)) = 0$ holds for any $v \in T_xM$ and any $V(0) \in T_xM$ satisfying $K(v, V(0)) = c$ at an arbitrary point $x \in M$. It follows that

$$\sigma(\dot\gamma, V) = 0 \quad \text{along } \gamma. \tag{3.13}$$

Taking the inner product of both sides of (3.7) with V, we have

$$\begin{aligned}\dot{\tilde\kappa}\kappa &= \tilde\kappa\dot\kappa - \tilde\kappa\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma, V\rangle \\ &= \tilde\kappa\dot\kappa - \tilde\kappa\langle\sigma(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, V)\rangle.\end{aligned}$$

Owing to (3.13), the above equation becomes

$$\dot{\tilde{\kappa}}\kappa = \tilde{\kappa}\dot{\kappa}. \tag{3.14}$$

Then the equation (3.8), together with (3.13) and (3.14), yields

$$(\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}) = \frac{\dot{\tilde{\kappa}}}{\tilde{\kappa}}\sigma(\dot{\gamma},\dot{\gamma}) = \frac{\dot{\kappa}}{\kappa}\sigma(\dot{\gamma},\dot{\gamma}),$$

and

$$(\bar{\nabla}_v\sigma)(v,v) = \frac{\dot{\kappa}(0)}{\kappa(0)}\sigma(v,v).$$

Changing v into $-v$, we get $(\bar{\nabla}_v\sigma)(v,v) = 0$. Thanks to Codazzi's equation in a space of constant curvature $(\bar{\nabla}_v\sigma)(u,w) = (\bar{\nabla}_u\sigma)(v,w)$, the immersion f is parallel.

Thus our immersion f is locally congruent to a parallel immersion $f_2 \circ f_1$ defined by (3.1) ([2, 8]). We here comment. According to the fundamental theorem of submanifolds, the fact that the second fundamental form is parallel implies the rigidity of our immersion. In fact, all connection forms of the normal bundle NM in $\widetilde{M}^{16+p}(\tilde{c})$ are uniquely determined by connection forms of M. So we can use the classification theorem of complete parallel submanifolds in a real space form as a local theorem. Note that our submanifold M is not necessarily complete.

Next, we shall show that the curve γ satisfying the hypothesis of our theorem is a Cayley circle by using a reduction to absurdity. So we assume that the curvature function κ is not constant. Then there exists some s_0 with $\dot{\kappa}(s_0) \neq 0$. Since $\kappa, \tilde{\kappa} > 0$, we find $\dot{\tilde{\kappa}}(s_0) \neq 0$ from (3.14). From the fact that $\bar{\nabla}\sigma = 0$ and (3.13) we can see that the equation (3.8) yields $\sigma(\dot{\gamma}(s_0),\dot{\gamma}(s_0)) = 0$. Moreover, we find that $\|\sigma(\dot{\gamma},\dot{\gamma})\|$ is constant along the curve γ. Indeed, by use of the fact that $\bar{\nabla}\sigma = 0$ and (3.13) we have

$$\begin{aligned}\frac{d}{ds}\|\sigma(\dot{\gamma},\dot{\gamma})\|^2 &= 2\langle(\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle \\ &\quad + 4\kappa\langle\sigma(V,\dot{\gamma}),\sigma(\dot{\gamma},\dot{\gamma})\rangle \\ &= 0.\end{aligned}$$

Thus we conclude $\sigma(v,v) = 0$ for an arbitrary unit vector $v \in T_xM$ at each point $x \in M$. Hence the immersion $f : M \to \widetilde{M}^{16+p}(\tilde{c})$ is totally geodesic. But it is known that our manifold M cannot be immersed into a real space form as a totally geodesic submanifold. Thus we have a contradiction, so that the curve γ is a Cayley circle.

Finally, we shall check the immersion $f = f_2 \circ f_1$ given by (3.1) satisfies the hypothesis of theorem in detail. Let $\gamma = \gamma(s)$ be a Cayley circle of curvature k (> 0), which satisfies $\nabla_{\dot{\gamma}}\dot{\gamma} = kV$ and $\nabla_{\dot{\gamma}}V = -k\dot{\gamma}$. Then, by the equation of Gauss we have

$$\begin{aligned}c &= \langle R(\dot{\gamma},V)V,\dot{\gamma}\rangle \\ &= \tilde{c} + \langle\sigma(\dot{\gamma},\dot{\gamma}),\sigma(V,V)\rangle - \|\sigma(\dot{\gamma},V)\|^2.\end{aligned} \tag{3.15}$$

On the other hand, since it is known that the immersion f is $\sqrt{c-\tilde{c}}$ (> 0)-isotropic (see for instance [7]), we have

$$\begin{aligned}&\langle\sigma(u,v),\sigma(w,y)\rangle + \langle\sigma(u,w),\sigma(y,v)\rangle \\ &+ \langle\sigma(u,y),\sigma(v,w)\rangle \\ &= (c-\tilde{c})\left(\langle u,v\rangle\langle w,y\rangle + \langle u,w\rangle\langle y,v\rangle + \langle u,y\rangle\langle v,w\rangle\right)\end{aligned}$$

for arbitrary u, v, w, y. Hence, particularly we have

$$2\|\sigma(\dot{\gamma},V)\|^2 + \langle\sigma(\dot{\gamma},\dot{\gamma}),\sigma(V,V)\rangle = c - \tilde{c}. \tag{3.16}$$

From (3.15) and (3.16) we obtain $\sigma(\dot{\gamma},V) = 0$.

The curve $f \circ \gamma$ satisfies $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = kV + \sigma(\dot{\gamma},\dot{\gamma})$, so that

$$\|\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\| = \sqrt{k^2 + \|\sigma(\dot{\gamma},\dot{\gamma})\|^2} = \sqrt{k^2 + c - \tilde{c}}.$$

Put

$$\widetilde{V} = \frac{1}{\sqrt{k^2+c-\tilde{c}}}\{kV + \sigma(\dot{\gamma},\dot{\gamma})\}.$$

Then

$$\begin{aligned}\widetilde{\nabla}_{\dot{\gamma}}\widetilde{V} &= \frac{1}{\sqrt{k^2+c-\tilde{c}}}\widetilde{\nabla}_{\dot{\gamma}}\{kV + \sigma(\dot{\gamma},\dot{\gamma})\} \\ &= \frac{1}{\sqrt{k^2+c-\tilde{c}}}\big\{k\left(\nabla_{\dot{\gamma}}V + \sigma(\dot{\gamma},V)\right) \\ &\qquad - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + D_{\dot{\gamma}}(\sigma(\dot{\gamma},\dot{\gamma}))\big\} \\ &= \frac{1}{\sqrt{k^2+c-\tilde{c}}}\big\{-k^2\dot{\gamma} - \|\sigma(\dot{\gamma},\dot{\gamma})\|^2\dot{\gamma} \\ &\qquad + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma}) + 2\sigma(\nabla_{\dot{\gamma}}\dot{\gamma},\dot{\gamma})\big\} \\ &= \frac{1}{\sqrt{k^2+c-\tilde{c}}}\big\{-(k^2+c-\tilde{c})\dot{\gamma} + 2k\sigma(V,\dot{\gamma})\big\} \\ &= -\sqrt{k^2+c-\tilde{c}}\;\dot{\gamma}.\end{aligned}$$

Here we have used the fact that $\sigma(\dot{\gamma},V) = 0$ and $A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} = \|\sigma(\dot{\gamma},\dot{\gamma})\|^2\dot{\gamma}$ which follows from the property of the isotropic immersion. Thus the curve $f \circ \gamma$ is a circle of curvature $\sqrt{k^2+c-\tilde{c}}$ (> 0), so that it is a plane curve in $\widetilde{M}^{16+p}(\tilde{c})$. $\square$

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