Two examples of nonconvex self-similar solution curves for a crystalline curvature flow

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Abstract: This note gives examples of nonconvex self-similar solutions for a crystalline curvature flow with an interfacial energy of which the Wulff shape is a regular triangle or a square.

Key words: Crystalline curvature; nonconvex self-similar shrinking curve; curve shortening flow equation; blow-up rate.

1. Introduction. In this note, we present two examples of homothetically shrinking nonconvex polygonal curves in the plane $\boldsymbol{R}^2$ moving under crystalline curvature flows. Such flows were originally defined by [3] and [9]. Since then several authors have considered its generalization; in a typical case the speed of motion of each edge is determined by a homogeneous function of some degree in its length.

Let us formulate the flow in this paragraph. Assume that an interfacial energy density γ is a convex function on $\boldsymbol{R}^2$ and satisfies $\gamma(r\cos\theta, r\sin\theta) = r\sigma(\theta)$ ($r \geq 0$, $\theta \in S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$) for some positive function $\sigma \in C(S^1)$. We consider the case where the Wulff shape of γ, $\mathcal{W}_\gamma = \bigcap_{\theta\in S^1}\{(x,y) \in \boldsymbol{R}^2 \mid x\cos\theta + y\sin\theta \leq \sigma(\theta)\}$, is a polygon. In this case, γ is called a *crystalline energy*, and we may express its Wulff shape as

$$\mathcal{W}_\gamma = \bigcap_{n=1}^{N} \{(x,y) \in \boldsymbol{R}^2 \mid x\cos\widetilde{\theta}_n + y\sin\widetilde{\theta}_n \leq \sigma(\widetilde{\theta}_n)\},$$

where $\widetilde{\theta}_n$ is the exterior normal angle of the n-th edge with $\widetilde{\theta}_n \in (\widetilde{\theta}_{n-1}, \widetilde{\theta}_{n-1}+\pi)$ for each n, and N is a number of edges ($N \geq 3$). Let $\mathcal{P}$ be a simple closed K-sided polygonal curve in $\boldsymbol{R}^2$, and label the vertices (x_k, y_k) ($k = 1, 2, \ldots, K$) in an anticlockwise order with $(x_0, y_0) = (x_K, y_K)$:

$$\mathcal{P} = \bigcup_{k=1}^{K} \mathcal{S}_k,$$

$$\mathcal{S}_k = \{(1-t)(x_{k-1}, y_{k-1}) + t(x_k, y_k) \mid 0 \leq t \leq 1\},$$

and let θ_k be the exterior normal angle of the k-th edge $\mathcal{S}_k$. We say that $\mathcal{P}$ is a K*-admissible curve* if the normal angles θ_k of all edges $\mathcal{S}_k$ belong to $\widetilde{\Theta}_\gamma = \{\widetilde{\theta}_1, \widetilde{\theta}_2, \ldots, \widetilde{\theta}_N\}$ and the angles of all adjacent edges in $\mathcal{P}$ are adjacent in $\widetilde{\Theta}_\gamma$ ($\subset S^1$). For each edge $\mathcal{S}_k$ a *crystalline curvature* is defined by $H_k = \chi_k \widetilde{l}_{n(k)}/l_k$, where l_k is the length of $\mathcal{S}_k$ and $\widetilde{l}_{n(k)}$ is the length of the n-th edge of $\mathcal{W}_\gamma$ satisfying $\widetilde{\theta}_n = \theta_k$. The quantity χ_k is a transition number, which takes -1 (resp., $+1$) if $\mathcal{P}$ is convex (resp., concave) at $\mathcal{S}_k$ in the outward normal direction $(\cos\theta_k, \sin\theta_k)$. Otherwise we set $\chi_k = 0$. Note that $\chi_k \equiv -1$ ($\forall k$) if $\mathcal{P}$ is a convex polygon and that the crystalline curvature of $\mathcal{W}_\gamma$ is -1 on each edge. Under a crystalline curvature flow each edge $\mathcal{S}_k$ keeps the same direction but moves in the outward normal direction with the velocity V_k determined by a homogeneous function of some degree $\alpha > 0$ in the crystalline curvature H_k:

$$V_k = \sigma(\theta_k)|H_k|^{\alpha-1}H_k \quad \text{on } \mathcal{S}_k \tag{1}$$

for $k = 1, 2, \ldots, K$. It is easy to show that if $K = N$ and $\mathcal{P}$ is homothety of $\partial\mathcal{W}_\gamma$, then $\mathcal{P}$ is a self-similar solution curve of (1); For $N = 3$ all admissible triangles are self-similar.

In this paper we give examples of a nonconvex self-similar solution curve shrinking to a point when the Wulff shape is a regular triangle or a regular square. Among other results we show that if $\alpha \in (0, 1)$, then such a nonconvex self-similar solution exists even if the motion is orientation-free, i.e.,

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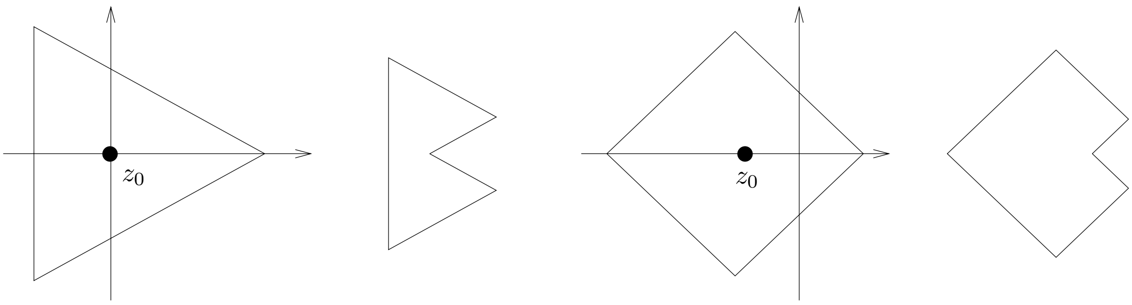

Fig. 1. Two examples in §2 and §3. From left to right: the Wulff triangle with $z_0 = 0$; the 5-admissible self-similar solution curve in the case $\alpha = 1$ and $c = (\sqrt{5} - 1)/2$; the Wulff square with $z_0 = -1/2$; and the 6-admissible self-similar solution curve in the case $\alpha = 1$ and $c = 2$.

$\sigma(\theta + \pi) = \sigma(\theta)$. This is a strong contrast to a motion by smooth interfacial energy density where the curve becomes convex in a finite time [4]. From our example it seems that general convexity statement in [5, Lemma 2 (i) (a)] is somewhat overstated. If $\alpha \geq 1$ and $\sigma(\theta + \pi) = \sigma(\theta)$, then a solution curve $\mathcal{P}_t$ of (1) with a K-admissible initial curve $\mathcal{P}_0$ converges to a single point or a K'-admissible curve with $K' < K$ as t tends to a finite time $T > 0$, and eventually $\mathcal{P}_t$ shrinks to a point at a finite time $T_* \geq T$ ([5]). Although it was stated that a solution becomes convex before it shrinks to a point ([5, Proposition 6]), a further investigation seems to be necessary to clarify in what generality such a convexity result hold.

For convex solution curves, on the other hand, detailed properties are known ([2, 5, 6, 7]). If $\alpha = 1$, $\sigma(\theta+\pi) = \sigma(\theta)$ and $N \geq 6$, then the only convex self-similar solution curve is a homothety of ∂W_γ ([8]). For a smooth interfacial energy density γ see, e.g., [1, 4].

2. The first example (case $N = 3$ and $K = 5$). We put $(p_n, q_n) = (z_0 + \cos(2n\pi/3), \sin(2n\pi/3))$ for $n = 0, 1, 2$ and $z_0 \in (-1, 1/2)$. Let a crystalline energy density γ be

$$\gamma(r\cos\theta, r\sin\theta) = r\sigma(\theta) = r \max_{n=0,1,2}\{p_n\cos\theta + q_n\sin\theta\}.$$

Then the Wulff shape of γ is a triangle with the vertices (p_n, q_n) $(n = 0, 1, 2)$:

$$\mathcal{W}_\gamma = \bigcap_{n=1}^{3}\{(x, y) \in \boldsymbol{R}^2 \mid x\cos\widetilde{\theta}_n + y\sin\widetilde{\theta}_n \leq \widetilde{h}_n\},$$

where $\widetilde{\theta}_n = \pi(2n - 1)/3$ and $\widetilde{h}_n = 1/2 + z_0\cos\widetilde{\theta}_n$. See Fig. 1 (far left). The length of each edge is $\widetilde{l}_n \equiv \sqrt{3}$ $(\forall n)$. We construct the 5-admissible curve $\mathcal{P} = \bigcup_{k=1}^{5}\mathcal{S}_k$ with the vertices (x_k, y_k) satisfying for $b > a > 0$ $(x_0, y_0) = (x_5, y_5) = (0, 0)$, $(x_1, y_1) = (\sqrt{3}a, a)/2$, $(x_2, y_2) = (\sqrt{3}(a-b), a+b)/2$, $(x_3, y_3) = (\sqrt{3}(a - b), -(a + b))/2$ and $(x_4, y_4) = (\sqrt{3}a, -a)/2$. The length of $\mathcal{S}_k$, $l_k = |(x_k - x_{k-1}, y_k - y_{k-1})|$, and its crystalline curvature satisfy $l_1 = l_5 = a$, $l_2 = l_4 = b$, $l_3 = a + b$ and $H_1 = H_5 = 0$, $H_2 = H_4 = -\sqrt{3}/b$, $H_3 = -\sqrt{3}/(a + b)$, respectively. Hence, by virtue of $\sigma(\theta_k) = \widetilde{h}_{n(k)}$ and $V_1 = V_5 = 0$, $V_2 = V_4 = \sqrt{3}\dot{a}/2$, $V_3 = \sqrt{3}(\dot{b} - \dot{a})/2$, evolution equations (1) are given as

$$\dot{a} = -\frac{1 + z_0}{\sqrt{3}}\left(\frac{\sqrt{3}}{b}\right)^\alpha, \quad \dot{b} - \dot{a} = -\frac{1 - 2z_0}{\sqrt{3}}\left(\frac{\sqrt{3}}{a + b}\right)^\alpha.$$

Here and hereafter $\widetilde{h}_{n(k)} = \widetilde{h}_n$ for $\widetilde{\theta}_n = \theta_k$, and $\dot{u}$ means du/dt. Putting $b - a = ac$, we have

$$\dot{c} = \frac{\sqrt{3}^{\alpha-1}(1 + z_0)}{a^{\alpha+1}(c + 2)^\alpha}\left(\frac{(c + 2)^\alpha c}{(c + 1)^\alpha} - \frac{1 - 2z_0}{1 + z_0}\right).$$

The *nonconvex* solution curve is self-similar if and only if $\dot{c} = 0$, that is

$$f(c, \alpha) := \frac{(c + 2)^\alpha c}{(c + 1)^\alpha} = \frac{1 - 2z_0}{1 + z_0}$$

holds. Then we have $\lim_{c\to+0} f(c, \alpha) = 0$ and $\lim_{c\to+\infty} f(c, \alpha) = +\infty$. If $0 < \alpha < 3 + 2\sqrt{2}$, then $\partial f(c, \alpha)/\partial c > 0$ holds for all $c > 0$. If $\alpha \geq 3 + 2\sqrt{2}$, then $\partial f(c, \alpha)/\partial c = 0$ holds only for $c = \big(\alpha - 3 \pm \sqrt{(\alpha - 3)^2 - 8}\big)/2 > 0$. Therefore, we have the following two cases:

Case $0 < \alpha \leq 3 + 2\sqrt{2}$. *For any $z_0 \in (-1, 1/2)$ there exists a unique $c > 0$ such that the solution is self-similar.* See Fig. 1 (left).

Case $\boldsymbol{\alpha > 3 + 2\sqrt{2}}$. *There exists two constants $-1 < z_- < z_+ < 1/2$ such that the following three cases hold*: (i) *For any $z_0 \in (-1, z_-) \cup (z_+, 1/2)$ there exists a unique $c > 0$ such that the solution is self-similar.* (ii) *For any $z_0 \in \{z_-, z_+\}$ there exist two positive constants c_1 and c_2 such that the solution is self-similar if and only if $c = c_1$ or c_2.* (iii) *For any $z_0 \in (z_-, z_+)$ there exist three positive constants c_1, c_2 and c_3 such that the solution is self-similar if and only if $c = c_1$, c_2 or c_3.*

3. The second example (case $\boldsymbol{N = 4}$ and $\boldsymbol{K = 6}$). We put $(p_n, q_n) = (z_0 + \cos(n\pi/2), \sin(n\pi/2))$ for $n = 0, 1, 2, 3$ and $z_0 \in (-1, 1)$. Let a crystalline energy density γ be

$$\begin{aligned}&\gamma(r\cos\theta, r\sin\theta)\\ &\quad = r\sigma(\theta) = r \max_{n=0,1,2,3}\{p_n\cos\theta + q_n\sin\theta\}.\end{aligned}$$

Then the Wulff shape of γ is a square with the vertices (p_n, q_n) $(n = 0, 1, 2, 3)$:

$$\mathcal{W}_\gamma = \bigcap_{n=1}^{4}\{(x, y) \in \mathbf{R}^2 \mid x\cos\widetilde{\theta}_n + y\sin\widetilde{\theta}_n \le \widetilde{h}_n\},$$

where $\widetilde{\theta}_n = \pi(2n-1)/4$ and $\widetilde{h}_n = 1/\sqrt{2} + z_0\cos\widetilde{\theta}_n$. See Fig. 1 (right). The length of each edge is $\widetilde{l}_n \equiv \sqrt{2}$ $(\forall n)$. We construct the 6-admissible curve $\mathcal{P} = \bigcup_{k=1}^{6}\mathcal{S}_k$ with the vertices (x_k, y_k) satisfying for $a > 0$ and $b > 0$ $(x_0, y_0) = (x_6, y_6) = (0, 0)$, $(x_1, y_1) = (a, a)/\sqrt{2}$, $(x_2, y_2) = (a - b, a + b)/\sqrt{2}$, $(x_3, y_3) = (-2b, 0)/\sqrt{2}$, $(x_4, y_4) = (a - b, -(a + b))/\sqrt{2}$ and $(x_5, y_5) = (a, -a)/\sqrt{2}$. The length of $\mathcal{S}_k$ and its crystalline curvature satisfy $l_1 = l_6 = a$, $l_2 = l_5 = b$, $l_3 = l_4 = a + b$ and $H_1 = H_6 = 0$, $H_2 = H_5 = -\sqrt{2}/b$, $H_3 = H_4 = -\sqrt{2}/(a + b)$, respectively. Hence, by virtue of $\sigma(\theta_k) = \widetilde{h}_{n(k)}$ and $V_1 = V_6 = 0$, $V_2 = V_5 = \dot{a}$, $V_3 = V_4 = \dot{b}$, evolution equations (1) are given as

$$\dot{a} = -\frac{1 + z_0}{\sqrt{2}}\left(\frac{\sqrt{2}}{b}\right)^{\alpha}, \quad \dot{b} = -\frac{1 - z_0}{\sqrt{2}}\left(\frac{\sqrt{2}}{a + b}\right)^{\alpha}.$$

Putting $b = ac$, we have

$$\text{(2)} \qquad \dot{c} = \frac{\sqrt{2}^{\alpha-1}(1 + z_0)}{a^{\alpha+1}(c + 1)^{\alpha}}\left(\frac{(c + 1)^{\alpha}}{c^{\alpha-1}} - \frac{1 - z_0}{1 + z_0}\right).$$

The *nonconvex* solution curve is self-similar if and only if $\dot{c} = 0$, that is

$$g(c, \alpha) := \frac{(c + 1)^{\alpha}}{c^{\alpha-1}} = \frac{1 - z_0}{1 + z_0}$$

holds.

Case $\boldsymbol{0 < \alpha < 1}$. Since $\lim_{c\to+0} g(c, \alpha) = 0$, $\lim_{c\to+\infty} g(c, \alpha) = +\infty$ and $\partial g/\partial c > 0$ $(\forall c > 0)$ hold, we have the following: *For any $z_0 \in (-1, 1)$ there exists a unique $c > 0$ such that the solution is self-similar.*

Case $\boldsymbol{\alpha = 1}$. Since $g(c, 1) = c + 1$, we have the following two cases: (i) *For any $z_0 \in [0, 1)$ and $c > 0$ the solution is not self-similar.* (ii) *For any $z_0 \in (-1, 0)$ the solution is self-similar if and only if $c = -2z_0/(1 + z_0) > 0$.* See Fig. 1 (far right).

Case $\boldsymbol{\alpha > 1}$. It holds that $\lim_{c\to+0} g(c, \alpha) = \lim_{c\to+\infty} g(c, \alpha) = +\infty$. Further, $\partial g(c, \alpha)/\partial c = 0$ holds if and only if $c = \alpha - 1$. Therefore, we have the following three cases: *Let $z_* = -\{\alpha^\alpha - (\alpha - 1)^{\alpha-1}\}/\{\alpha^\alpha + (\alpha - 1)^{\alpha-1}\} \in (-1, 0)$.* (i) *For any $z_0 \in (z_*, 1)$ and $c > 0$ the solution is not self-similar.* (ii) *For $z_0 = z_*$ the solution is self-similar if and only if $c = \alpha - 1 > 0$.* (iii) *For any $z_0 \in (-1, z_*)$ there exist two positive constants c_1 and c_2 such that the solution is self-similar if and only if $c = c_1$ or c_2.*

Remark in case $\boldsymbol{\alpha = 1}$. All convex solutions are self-similar. On the other hand, when $z_0 \in (-1, 0)$ and $c_0 = c(0) \in (0, -2z_0/(1 + z_0))$, the nonconvex solution *shrinks to a single point* and $\max_k 1/l_k$ blows up to infinity as t tends to a finite time T with its rate being *faster than the self-similar rate.* Indeed, from (2), $c < -2z_0/(1+z_0)$ implies $\dot{c} < 0$. Hence, $b(T) = 0$ $(c_0 a(t) \ge b(t) > 0)$ holds, and so $a(T) = 0$ holds since no degenerate pinching occurs [5, Lemma 2 (iii)], which implies a single point extinction. Also, the enclosed area $A = (2a + b)b$ satisfies $A(t) = 4(T - t)$. Therefore, we obtain the estimate $b(t) \le C(T - t)^{d_0}$ for some $C > 0$ since $\dot{b} = -(1 - z_0)(c + 2)b/\{(c + 1)A\} \le -d_0 b/(T - t)$ holds. Here $d_0 = (1 - z_0)(c_0 + 2)/\{4(c_0 + 1)\}$, which satisfies $d_0 \in (1/2, 1)$. Hence $b(t)$ never does admit the self-similar rate $\sqrt{T - t}$. Furthermore, $a(t) \ge 2C^{-1}(T - t)^{1-d_0} - 2^{-1}C(T - t)^{d_0}$ holds, which implies the isoperimetric ratio $L(t)^2/A(t)$ diverges to infinity as t tends to T. Here $L(t) = 2(a + b)$ is the total length.

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