On an infinite convolution product of measures

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Abstract: We prove that infinite convolution products of complex probability measures with bounded total variation converge to a hyperfunction under a weak assumption on supports.

Key words: Convolution product; hyperfunction.

This paper concerns infinite convolution products of complex probability Radon measures with bounded total variation on $\mathbf{R}^n$, and we shall prove that they converge to a hyperfunction under a weak assumption on supports. This kind of convergence problem is studied in connection with wavelet construction (see [DD], [L1]). The author was inspired by a talk of W. Lawton [L2] to establish the result presented here, which is a hyperfunction analogue of a theorem of [DD] and will give us the most general framework for convergence of infinite convolution products of complex probability measures. The proof is an exercise of the theory of hyperfunctions.

1. Main result. For a complex Radon measure u on $\mathbf{R}^n$ with compact support, $||u||$ denotes its total variation, and supp u its support. $u * v$ denotes the convolution product of u and v.

For $r > 0$, $B(r)$ denotes the closed ball centered at the origin with radius r.

For a compact subset K of $\mathbf{R}^n$, let us consider $\mathcal{B}[K]$, the space of Sato hyperfunctions on $\mathbf{R}^n$ with support contained in K. Recall that this space is the topological dual of the space $\mathcal{A}(K)$ of analytic functions on K and is a FS space endowed with the strong dual topology (see for example [M], [S]).

Theorem 1. *Let $\{u_\nu\}_{\nu\geq 0}$ be a sequence of complex measures with compact support. Assume the following*:

(1) $u_\nu(\mathbf{R}^n) = 1$ *for all* ν;
(2) *The total variations* $\|u_\nu\|$, $\nu \geq 0$, *are bounded (i.e.,* $\|u_\nu\| < C$ *for all* ν, *with some* $C \geq 0$ *independent of* ν);
(3) $\operatorname{supp} u_\nu \subset B(r_\nu)$ *with* $r_\nu > 0$, *and*

$$\sum_{\nu=0}^{\infty} r_\nu < \infty.$$

Then, for $R = \sum_{\nu\geq 0} r_\nu$, *the convolution product* $u_0 * \cdots * u_\nu$ $(\nu \to \infty)$ *converges to a hyperfunction supported in* $B(R)$.

Note that the convergence is realized in the topology of $\mathcal{B}[B(R)]$.

Remark. Let C_ν be the total variation of u_ν. We can replace (2) and (3) by the following assumption:

$$\sum_{\nu=0}^{\infty} C_\nu r_\nu < \infty \quad \text{and} \quad \sum_{\nu=0}^{\infty} r_\nu < \infty.$$

If we also assume (in (3)) that $r_\nu \leq C\alpha^\nu$ for some $\alpha < 1$ and $C > 0, u_0 * \cdots * u_\nu$ converges to a distribution (in the topology of the space of distributions). This result was first proved by Deslauriers and Dubuc [DD]. Lawton [L1] gave another proof based on the Taylor expansion, which is applied also to convolution products of measures on a non-commutative Lie group.

2. Proof of Theorem 1. Let $f_\nu(\zeta)$ be the Fourier-Borel transform of u_ν:

$$f_\nu(\zeta) = \int e^{-ix\zeta} u_\nu(x)dx.$$

Then, by (1),

$$f_\nu(\zeta) = 1 + \int (e^{-ix\zeta} - 1)u_\nu(x)dx,$$

and, by (2),

$$\left| \int (e^{-ix\zeta} - 1)u(x)dx \right| \leq C \sup_{|x|\leq r_\nu} |e^{-ix\zeta} - 1| \leq C r_\nu |\zeta| e^{r_\nu |\operatorname{Im}\zeta|}.$$

Therefore we have $f_\nu(\zeta) = 1 + g_\nu(\zeta)$ and

$$|g_\nu(\zeta)| \leq C r_\nu |\zeta| e^{r_\nu |\operatorname{Im}\zeta|}. \tag{2.1}$$

By (3), we see that $\sum_{\nu\geq 0} |g_\nu(\zeta)|$ converges locally

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uniformly in ζ, and hence

$$\prod_{\nu=0}^{\infty} f_\nu(\zeta)$$

converges locally uniformly to an entire holomorphic function $F(\zeta)$.

By (3), for any $\varepsilon > 0$, there exists a non-negative integer N such that $\sum_{\nu \geq N} r_\nu \leq \varepsilon/2C$. Then there is a positive number C_ε such that

$$(1 + Cr_0|\zeta|) \cdots (1 + Cr_{N-1}|\zeta|) \leq C_\varepsilon e^{\varepsilon|\zeta|/2}, \ \zeta \in \mathbf{C}^n.$$

Since we have

$$|f_\nu(\zeta)| \leq (1 + Cr_\nu|\zeta|)e^{r_\nu |\operatorname{Im}\zeta|}$$

by (2.1), it follows that

$$\prod_{\nu=0}^{\infty} |f_\nu(\zeta)| \leq C_\varepsilon e^{\varepsilon|\zeta| + R|\operatorname{Im}\zeta|}, \ \zeta \in \mathbf{C}^n, \tag{2.2}$$

where $R = \sum_{\nu \geq 0} r_\nu$; therefore $F(\zeta)$ is estimated by the second member of (2.2).

By the Paley-Wiener-Ehrenpreis theorem for hyperfunctions (see for example [M]), F is the Fourier-Borel transform of a hyperfunction β with support in $B(R)$. Let

$$F_k(\zeta) = f_0(\zeta) \cdots f_k(\zeta).$$

For any $\varepsilon > 0$ and any $\eta > 0$, we have

$$|F_k(\zeta) - F(\zeta)| \leq \eta e^{\varepsilon|\zeta| + R|\operatorname{Im}\zeta|}, \tag{2.3}$$

if k is sufficiently large. In fact, from (2.1) and (2.2), we have

$$\begin{aligned} |F_k(\zeta) - F(\zeta)| &\leq \sum_{\nu=k}^{\infty} |F_\nu(\zeta) - F_{\nu+1}(\zeta)| \\ &= \sum_{\nu=k}^{\infty} |F_\nu(\zeta) g_{\nu+1}(\zeta)| \\ &\leq C_\varepsilon \exp\{(\varepsilon + \sup\{r_\nu \mid \nu > k\})|\zeta| \\ &\quad + R|\operatorname{Im}\zeta|\} \sum_{\nu=k+1}^{\infty} r_\nu, \end{aligned}$$

with some $C_\varepsilon > 0$, for any $\varepsilon > 0$. The Fourier-Borel transform of hyperfunctions with compact support yields an isomorphism of FS spaces $\mathcal{B}[B(R)] \cong \operatorname{Exp}(R)$, where the second member denotes the vector space endowed with natural Fréchet topology, consisting of entire holomorphic functions $F(\zeta)$ such that

$$|F(\zeta)| \leq C_\varepsilon e^{\varepsilon|\zeta| + R|\operatorname{Im}\zeta|}, \ \zeta \in \mathbf{C}^n,$$

with some $C_\varepsilon > 0$, for any $\varepsilon > 0$ (see Theorem 6.4.5 of [M]). By (2.3), we have $F_k \to F$ in $\operatorname{Exp}(R)$. This implies that

$$u_0 * \cdots * u_k \to \beta, \quad k \to \infty,$$

in the topology of $\mathcal{B}[B(R)]$.

This completes the proof. □

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