Homotopy groups of the homogeneous spaces F_4/G_2, $F_4/\operatorname{Spin}(9)$ and E_6/F_4

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Abstract: In the paper we calculate 2-primary components of homotopy groups of the hmogeneous spaces $F_4/G_2, F_4/\mathrm{Spin}(9)$ and E_6/F_4.

Key words: Homotopy group; homogeneous space; exceptional Lie group.

1. Introduction. Let G_2, F_4 and E_6 be the compact, connected, simply connected, simple, exceptional Lie groups of rank 2, 4 and 6 respectively. We consider the homogeneous spaces F_4/G_2, $F_4/\operatorname{Spin}(9) = \Pi$ and E_6/F_4, where Π denotes the Cayley projective plane. We denote by $\pi_i(X : p)$ the $p-$primary component of $\pi_i(X)$. In this paper we calculate homotopy groups $\pi_i(F_4/G_2 : 2)$, $\pi_i(\Pi : 2)$ ans $\pi_i(E_6/F_4 : 2)$ for $i \leq 45$, $i \leq 38$ and $i \leq 30$ respectively. The caleulations of $\pi_i(F_4/G_2 : 2)$ and $\pi_i(\Pi : 2)$ will be done by making use of the homotopy exact sequences associated with the 2-local fibration

$$S^{15} \longrightarrow F_4/G_2 \longrightarrow S^{23}$$

and the fibration

$$S^7 \longrightarrow \Omega\Pi \longrightarrow \Omega S^{23}$$

given by Davis and Mahowald [3]. The calculation of $\pi_i(E_6/F_4 : 2)$ will be done by making use of the 2-local fibration

$$X \longrightarrow S^9 \longrightarrow E_6/F_4,$$

where X is the homotopy fibre of the natural inclusion of S^9 in E_6/F_4. To determine the group extension we use the following theorem which is proved by Mimura and Toda [10].

Theorem 1.1 (Theorem 2.1 of [10]). *Let (X, p, B) be a fibration with the fibre $F(= p^{-1}(*))$ and Δ the boundary homomorphism of the homotopy exact sequence associated with the fibration. Assume that $\alpha \in \pi_{i+1}(B)$, $\beta \in \pi_j(S^i)$ and $\gamma \in \pi_k(S^j)$ satisfy the conditions $(\Delta(\alpha))\beta = 0$ and $\beta\gamma = 0$. For an arbitrary element δ of Toda bracket $\{\Delta(\alpha), \beta, \gamma\} \subset \pi_{k+1}(F)$, there exists an element $\varepsilon \in \pi_{j+1}(X)$ such that*

$$p_*\varepsilon = \alpha E\beta, \quad i_*\delta = \varepsilon E\gamma,$$

where $i : F \to X$ is an inclusion map.

The notations and the terminologies in [6], [7], [9], [10], [11], [12], [14], [16] will be freely used in the present paper, and we also omit for simplicity the notation $\circ$ indicating composition.

The results in the present pater shall be used to deduce $\pi_i(F_4)$ adn $\pi_i(E_6)$ in the forthcoming paper.

2. Homotopy groups of F_4/G_2. We consider the 2-local fibration

$$S^{15} \xrightarrow{i} F_4/G_2 \xrightarrow{p} S^{23}$$

which is given by Davis and Mahowald [3]. Then we have the homotopy exact sequence

$$\cdots \longrightarrow \pi_{i+1}(S^{23} : 2) \xrightarrow{\Delta i+1} \pi_i(S^{15} : 2)$$
$$\xrightarrow{i_*} \pi_i(F_4/G_2 : 2) \xrightarrow{p_*} \pi_i(S^{23} : 2) \xrightarrow{\Delta_i} \cdots$$

associated with the above 2-local fibration. This exact sequence induces an exact one:

(1) $0 \to \mathrm{Coker}\Delta_{i+1} \xrightarrow{i_*} \pi_i(F_4/G_2 : 2) \xrightarrow{p_*} \mathrm{Ker}\Delta_i \to 0.$

Since $H^*(F_4/G_2; \mathbf{Z}_2) \cong \wedge(x_{15}, Sq^8 x_{15})$ ([1]), we have the 2-local equivalence

$$F_4/G_2 \underset{2}{\simeq} S^{15} \underset{\sigma_{15}}{\cup} e^{23} \cup e^{38}.$$

Here we have the formulas

(2) $\Delta_{23}(\iota_{23}) = \sigma_{15}$ and $\Delta_i(E\alpha) = \sigma_{15}\alpha$

where ι_{23} is the homotopy class of the identity map of S^{23} and $\alpha \in \pi_{i-1}(S^{22} : 2)$. By making use of the formula (2), we calculate the kernel and the cokernel

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of the boundary homomorphism $\Delta_i : \pi_i(S^{23} : 2) \to \pi_{i-1}(S^{15} : 2)$. The results are stated in the following.

Lemma 2.1. *We have the following table of the kernel and the cokernel of* Δ_i.

i	23	24	25	26	27	28	29
$\mathrm{Ker}\Delta_i$	∞	0	0	8	0	0	2
$\mathrm{Coker}\Delta_{i+1}$	2	$(2)^2$	2	8	0	0	2

30	31	32	33	34	35
4	$(2)^2$	$(2)^2$	0	8	0
$32+2$	$(2)^2$	$(2)^4$	$(8)^2+2$	$8+(2)^2$	8

36	37	38	39	40
0	2	$16+2$	2	$(2)^2$
$(2)^2$	$8+(2)^3$	$16+8+(2)^3$	$(2)^4$	$(2)^3$

41	42	43	44	45
2	$8+2$	8	$(2)^2$	$(2)^2$
$8+(2)^2$	8	2	2	$8+(2)^2$

Here an integer n indicates a cyclic group $\mathbf{Z}_n$ of order n, the symbol ∞ an infinite eyelie group $\mathbf{Z}$, the symbol + the direct sum of groups and $(n)^k$ indicates the direct sum of k-copies of $\mathbf{Z}_n$.

Let us state our first main result.

Theorem 2.2. *We have the following table of the homotopy groups* $\pi_i(F_4/G_2 : 2)$ *for* $i \leq 45$.

i	$i \leq 14$	15	16	17	18
$\pi_i(F_4/G_2 : 2)$	0	∞	2	2	8

19, 20	21	22	23	24	25	26
0	2	0	$\infty+2$	$(2)^2$	2	64

27, 28	29	30	31	32	33
0	$(2)^2$	$128+2$	$(2)^4$	$(2)^6$	$(8)^2+2$

34	35	36	37
$64+(2)^2$	8	$(2)^2$	$8+4+(2)^2$

38	39	40	41	42
$256+8+(2)^4$	$(2)^5$	$(2)^5$	$8+4+2$	$64+2$

43	44	45
$8+2$	$(2)^3$	$8+(2)^4$

Proof. From Lemma 2.1, it follows that the homomorphisms $i_* : \mathrm{Coker}\,\Delta_{i+1} \to \pi_i(F_4/G_2 : 2)$ are isomorphisms for $i \leq 22$ and $i = 24,\ 25,\ 27,\ 28,\ 33,\ 35,\ 36$.

We remark that $\pi_{27}(F_4/G_2 : 2) = \pi_{28}(F_4/G_2 : 2) = 0$.

Consider the case $i = 26$. By Lemma 2.1 and the exact sequence (1), we have an exact sequence

$$0 \longrightarrow \mathbf{Z}_8 \xrightarrow{i_*} \pi_{26}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_8 \longrightarrow 0,$$

where the first $\mathbf{Z}_8$ is generated by ζ_{15} and the second $\mathbf{Z}_8$ is generated by ν_{23}. For the Toda bracket, we have

$$\{\sigma_{15}, \nu_{23}, 8\iota_{25}\} \ni x\zeta_{15}$$

for some old integer x. By Theorem 1.1, these exists an element $[\nu_{23}] \in \pi_{26}(F_4/G_2 : 2)$ such that

$$p_*([\nu_{23}]) = \nu_{23} \quad \text{and} \quad i_*(x\zeta_{15}) = 8[\nu_{23}].$$

Therefore we obtain $\pi_{26}(F_4/G_2 : 2) = \{[\nu_{23}]\} \cong \mathbf{Z}_{64}$.

For $i = 30,\ 34,\ 37,\ 38,\ 41,\ 42$, we obtain the results of $\pi_i(F_4/G_2 : 2)$ by an argument similar to the case $i = 26$.

Consider the case $i = 29$. By Lemma 2.1 and the exact sequence (1), we have an exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \xrightarrow{i_*} \pi_{29}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \longrightarrow 0,$$

where the first $\mathbf{Z}_2$ is generated by κ_{15} and the second $\mathbf{Z}_2$ is generated by ν_{23}^2. We consider $[\nu_{23}]\nu_{26}$. We have

$$\begin{aligned} 2([\nu_{23}]\nu_{26}) &= [\nu_{23}]E^{23}\nu' && \text{by (5.5) of [16]} \\ &\in [\nu_{23}]\{\eta_{26}, 2\iota_{27}, \eta_{27}\} && \text{by the definition of } \nu'\text{([16])} \\ &\subset \{[\nu_{23}]\eta_{26}, 2\iota_{27}, \eta_{27}\}. \end{aligned}$$

Since $\pi_{27}(F_4/G_2 : 2) = \pi_{28}(F_4/G_2 : 2) = 0$, we have $\{[\nu_{23}]\eta_{26}, 2\iota_{27}, \eta_{27}\} = \{0, 2\iota_{27}, \eta_{27}\} \equiv 0 \bmod 0$. Therefore we have

$$2([\nu_{23}]\nu_{26}) = 0.$$

Moreover we have

$$p_*([\nu_{23}]\nu_{26}) = (p_*\nu_{23}])\nu_{26} = \nu_{23}^2.$$

This implies that the above sequence splits.

For $i = 31,\ 32,\ 39,\ 40,\ 43,\ 44,\ 45$, we obtain the results of $\pi_i(F_4/G_2 : 2)$ by an argument similar to the case $i = 29$. □

3. Homotopy groups of $\Pi = F_4/\mathrm{Spin}(9)$.

We consider the fibration

$$S^7 \xrightarrow{i} \Omega\Pi \xrightarrow{p} \Omega S^{23}$$

which is given by Davis and Mahowald [3]. Then we have the homotopy exact sequence

$$\cdots \longrightarrow \pi_{i+1}(\Omega S^{23}:2) \xrightarrow{\Delta_{i+1}} \pi_i(S^7:2)$$
$$\xrightarrow{i_*} \pi_i(\Omega\Pi:2) \xrightarrow{p_*} \pi_i(\Omega S^{23}:2) \xrightarrow{\Delta_i} \cdots$$

associated with the above fibration. This exact sequence induces an exact one:

(3) $0 \to \mathrm{Coker}\Delta_{i+1} \overset{i_*}{\to} \pi_i(\Omega\Pi:2) \overset{p_*}{\to} \mathrm{Ker}\Delta_i \to 0.$

By Davis-Mahowald [3] and Mimura [8], we have the 2-local equivalence

$$\Omega\Pi \underset{2}{\simeq} S^7 \underset{\sigma'\sigma_{14}}{\cup} e^{22} \cup e^{29} \cup \cdots .$$

Here we have the formulas

(4) $\Delta_{22}\,\mathrm{ad}(\iota_{23}) = \sigma'\sigma_{14}$ and $\Delta_i\,\mathrm{ad}(E^2\alpha) = \sigma'\sigma_{14}\alpha$,

where $\mathrm{ad}: \pi_{23}(S^{23}:2) \to \pi_{22}(\Omega S^{23}:2)$ is the adjoint isomorphism and $\alpha \in \pi_{i-2}(S^{21}:2)$. By making use of the formula (4), we calculate the kernel and the cokernel of the boundary homomorphism $\Delta_i : \pi_i(\Omega S^{23}:2) \to \pi_{i-1}(S^7:2)$. The results are stated in the following.

Lemma 3.1. *We have the following table of the kernel and the cokernel of* Δ_i.

i	22	23	24	25
Ker Δ_i	∞	0	0	8
Coker Δ_{i+1}	$8+(2)^2$	$(2)^3$	$(2)^4$	$8+2$

26	27	28	29	30	31
0	0	2	16	$(2)^2$	$(2)^2$
$8+2$	8	$(2)^2$	$8+(2)^3$	$(8)^2+(2)^3$	$(2)^6$

32	33	34	35
0	8	0	0
$8+4+(2)^3$	$8+(2)^6$	$(8)^2+2$	$(2)^4$

36	37
$(2)^2$	$32+2$
$8+(2)^4$	$(8)^2+(2)^2$

Theorem 3.2. *We have the following table of the homotopy groups* $\pi_i(\Pi:2)$ *for* $i \le 38$.

i	$i \le 7$	8	9	10	11	12, 13
$\pi_i(\Pi:2)$	0	∞	2	2	8	0

14	15	16	17	18	19	20	21
2	8	$(2)^3$	$(2)^4$	$8+2$	$8+2$	0	2

22	23	24	25	26
4	$\infty+8+(2)^2$	$(2)^3$	$(2)^4$	$64+2$

27	28	29	30	31
$8+2$	8	$(2)^3$	$128+(2)^3$	$(8)^2+(2)^5$

32	33	34	35	36
$(2)^8$	$8+4+(2)^3$	$64+(2)^6$	$(8)^2+2$	$(2)^4$

37	38
$16+4+(2)^3$	$256+8+(2)^3$

Proof. From Lemma 3.1, it follows that the homomorphism $i_* : \mathrm{Coker}\,\Delta_{i+1} \to \pi_i(\Omega\Pi:2)$ are isomorphisms for $i \le 21$ and $i = 23,\ 24,\ 26,\ 27,\ 32,\ 34,\ 35$.

Consider the case $i = 25$. By Lemma 3.1 and the exact sequence (3), we have an exact sequence

$$0 \longrightarrow \mathbf{Z}_8 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{25}(\Omega\Pi:2) \xrightarrow{p_*} \mathbf{Z}_8 \longrightarrow 0.$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_2$ is gencrated by $\zeta_7\sigma_{18}$, $\eta_7\bar{\mu}_8$ and $\mathbf{Z}_8$ is generated by $\mathrm{ad}(\nu_{23})$. For the Toda bracket, we have

$$\{\sigma'\sigma_{14}, \nu_{21}, 8\iota_{24}\} \ni x\zeta_7\sigma_{18}$$

for some odd integer x. By Theorem 1.1, there exists an element $[\nu_{23}] \in \pi_{25}(\Omega\Pi:2)$ such that

$$p_*([\nu_{23}]) = \mathrm{ad}(\nu_{23}) \quad \text{and} \quad i_*(x''\zeta_7\sigma_{18}) = 8[\nu_{23}].$$

Therefore we have $\pi_{25}(\Omega\Pi:2) \cong \mathbf{Z}_{64} \oplus \mathbf{Z}_2$.

For $i = 29,\ 33,\ 36,\ 37$, we obtain the results of $\pi_i(\Omega\Pi:2)$ by an argument similar to the case $i = 25$.

Consider the case $i = 28$. By Lemma 3.1 and the exact sequence (3), we have an exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{28}(\Omega\Pi:2) \xrightarrow{p_*} \mathbf{Z}_2 \longrightarrow 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\eta_7\bar{\kappa}_8$, $\sigma'\kappa_{14}$ and $\mathbf{Z}_2$ is generated by $\mathrm{ad}(\nu_{23}^2)$. For the Toda bracket, we have

$$\{\sigma'\sigma_{14}, \nu_{21}^2, 2\iota_{27}\} = 0.$$

By Theorem 1.1, there exists an element $\varepsilon \in \pi_{28}(\Omega\Pi:2)$ such that $p_*(\varepsilon) = \mathrm{ad}(\nu_{23}^2)$ and $2\varepsilon = 0$. Since $[\nu_{23}]\nu_{25} - \varepsilon \subset \mathrm{Im}\,i_*$, we have $2[\nu_{23}]\nu_{25} = 0$. Therefore we can choose $[\nu_{23}]\nu_{25}$ as a generator. Then we obtain the required result.

For $i = 30,\ 31$, we obtain the resuits of $\pi_i(\Omega\Pi:2)$ by an argument similar to the case $i = 28$. □

4. Homotopy groups of E_6/F_4. Since $H^*(E_6/F_4;\mathbf{Z}_2) \cong \wedge(x_9, Sq^8x_9)$, we have

$$E_6/F_4 \underset{2}{\simeq} S^9 \underset{\sigma_9}{\cup} e^{17} \cup e^{26}.$$

Let X denote the homotopy fibre of the inclusion of S^9 in E_6/F_4. Then we have

$$H^*(X;\mathbf{Z}_2) \cong \wedge(x_{16}, x_{32}, \ldots, x_{16\cdot 2^i}, \ldots)$$

and

$$X \underset{2}{\simeq} S^{16} \cup e^{32} \cup e^{48} \cup \cdots .$$

Therefore for $i \leq 30$, we have the homotopy exact sequence

$$\cdots \longrightarrow \pi_i(S^{16}:2) \xrightarrow{\sigma_{9*}} \pi_i(S^9:2) \longrightarrow$$
$$\pi_i(E_6/F_4:2) \longrightarrow \pi_{i-1}(S^{16}:2) \xrightarrow{\sigma_{9*}} \cdots .$$

We consider the short exact sequence

$$0 \to \mathrm{Coker}\,\sigma_{9*} \to \pi_i(E_6/F_4:2) \to \mathrm{Ker}\,\sigma_{9*} \to 0.$$

For the case $i = 16$, we have Ker $\sigma_{9*} \cong \mathbf{Z}$. For the other values of i ($i \leq 30$), the homomorphisms $\sigma_{9*} : \pi_i(S^{16}:2) \to \pi_i(S^9:2)$ are monomorphism. Therefore we can calculate $\pi_i(E_6/F_4:2)$ easily.

Theorem 4.1. *We have the following table of the homotopy groups $\pi_i(E_6/F_4:2)$ for $i \leq 30$.*

i	$i \leq 7$	8	9	10	11	12
$\pi_i(E_6/F_4:2)$	0	0	∞	2	2	8

13	14	15	16	17	18	19
0	0	2	0	$\infty + (2)^2$	$(2)^3$	2

20	21	22	23	24	25	26	27
$8+2$	0	0	4	$16+2$	2	$(2)^3$	2

28	29	30
$8+2$	8	2

Remark. To calculate $\pi_i(E_6/F_4:2)$ further, we need to determine the homotopy type of X.

References

[1] Borel, A.: Sur I'homologie et la cohomologie des groupes de Lie compacts connexes. Amer. J. Math., **76**, 273–342 (1954).

[2] Conlon, L.: An application of the Bott suspension map to the topology of EIV. Pacific J. Math., **19**, 411–428 (1966).

[3] Davis, D.M., and Mahowald, M.: Three contributions to the homotopy theory of the exceptional Lie groups G_2 and F_4. J. Math. Soc. Japan, **43**, 661–671 (1991).

[4] Hirato, Y.: Homotopy groups of F_4/G_2. Master's thesis of Shinshu Univ. (1996) (in Japanese).

[5] Hirato, Y., and Mukai, J.: Some Toda bracket in $\pi^S_{26}(S^0)$. J. Math. Okayama Univ. (to appear).

[6] Mimura, M.: On the generalized Hopf homomorphism and the higher composition, Part I. J. Math. Kyoto Univ., **4**, 171–190 (1964).

[7] Mimura, M.: On the generalized Hopf homomorphism and the higher composition, Part II. J. Math. Kyoto Univ. **4**, 301–326 (1965).

[8] Mimura, M.: The Homotopy groups of Lie groups of low rank, J. Math. Kyoto Univ., **6**, 131–176 (1967).

[9] Mimura, M., Mori, M., and Oda, N.: Determination of 2-components of the 23 and 24-stems in homotopy groups of spheres. Mem. Fac. Sci. Kyushu Univ. Ser. A, **29**, 1–42 (1975).

[10] Mimura, M., and Toda, H.: Homotopy groups of $SU(3)$, $SU(4)$ and $Sp(2)$. J. Math. Kyoto Univ., **3**, 217–250 (1964).

[11] Mimura, M., and Toda, H.: The $(n+20)$-th homotopy groups of n-spheres. J. Math. Kyoto Univ., **3**, 37–58 (1963).

[12] Mukai, J.: On the stable homotopy of a $\mathbf{Z}_2$-Moore space, Osaka J. Math., **6**, 63–91 (1969).

[13] Oda, N.: On the 2-components of the unstable homotopy groups of spheres. II. Proc. Japan Acad., **53A**, 215–218 (1977).

[14] Oda, N.: Unstable homotopy groups of spheres. Bull. Inst. Adv. Res. Fukuoka Univ., **44**, 49–152 (1979).

[15] Oda, N.: Some Toda brackets and the relations of H.Ōshima in the homotopy groups of spheres. Fukuoka Univ. Sei. Rep., **15**, 5–11 (1985).

[16] Toda, H.: Composition Methods in Homotopy Groups of Spheres. Ann. of Math. Studies, no. 49, Princeton (1962).