A special divisor on a double covering of a compact Riemann surface

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Introduction. By a curve we shall mean a connected compact Riemann surface. Let $l(D):=\dim H^0(C, \mathcal{O}(D))$ and $c(D):=\deg D - 2l(D) + 2$ for a divisor D on a curve C of genus $g \geq 2$. It is easy to check $c(D) = c(K_C - D)$ by using the Riemann-Roch theorem. Here K_C is a canonical divisor of C. Clifford's theorem states that $c(D) \geq 0$ if $l(D) > 0$ and $l(K_C - D) > 0$; moreover if there is a divisor D such that $l(D) > 0$, $c(D) = 0$ but $D \not\sim 0$ and $D \not\sim K_C$, then C is a hyperelliptic curve. In other words, we can say that a curve has a special divisor D with $c(D) = 0$ if and only if it is 0-hyperelliptic, where a special divisor D means $2 \leq \deg D \leq g - 1$ and $l(D) > 0$ (so $l(K_C - D) > 0$), a g'-hyperelliptic curve means a curve which is a double covering of a curve of genus g'.

We would like to classify double coverings of a curve with small genus by the index cliff $(C):=\min\{c(D) : D \text{ is a special divisor on } C, l(D) \geq 2\}$. We show that a curve having a special divisor D with small $c(D)$ is g'-hyperelliptic with $g' \leq c(D)/2$ [Theorem 1], and conversely, a g'-hyperelliptic curve has a special divisor D with $c(D) = 2g'$ [Theorem 2]. In particular, we obtain a necessary and sufficient condition for a curve to be 1-hyperelliptic [Corollary].

Main results. Theorem 1. *Let C be a curve of genus $g \geq 2$. Assume that there is an effective base-point-free divisor D with $\deg D \leq g - 1$, $l(D) \geq c(D) + 3$. Then $c(D)$ is even and C is a g'-hyperelliptic curve with some $g' \leq c(D)/2$, $g \geq 6g' + 5$.*

Proof. To give a proof of this theorem, we use the following inequality of Castelnuovo [1] (p.116):

Lemma. *Let C' be a curve that admits a birational mapping onto a (not necessarily smooth) non-degenerate curve (i.e., a curve not contained in any hyperplane of the projective n-space $\mathbf{P}^n$) of degree d' in $\mathbf{P}^n$. Then the genus of C' satisfies the inequality $g(C') \leq m(m-1)(n-1)/2 + m\varepsilon$, where $m:=[(d'-1)/(n-1)]$ and $\varepsilon:=(d'-1) - m(n-1)$.*

Under the hypothesis of Theorem 1, $c:=c(D) \geq 0$ by Clifford's theorem. Since D is base-point-free, we can define a map $\varphi : C \to \mathbf{P}^n$ associated with D. $\varphi(C)$ is non-degenerate by construction. Let C' be the normalization of $\varphi(C)$, $\nu : C' \to \varphi(C)$ the normalization map and $\tilde{\varphi} : C \to C'$ the induced map of φ. Put $d:=\deg D$, $n:=l(D) - 1$, $g':=g(C')$ and $d':=\deg \varphi(C)$. Then $c = d - 2n$ and $n \geq c + 2$.

Claim. $\deg \varphi = 2$.

1. If $\deg \varphi \geq 3$, then $d' \leq d/3$ and $n - d' \geq n - d/3 = (n-c)/3 > 0$. The above lemma implies $g = 0$ so $C' = \mathbf{P}^1$. Put $\mathcal{O}_{\mathbf{P}^1}(N):=\nu^*\mathcal{O}_{\mathbf{P}^n}(1)$. Since $\varphi(C)$ is nondegenerate, $\nu^* : \Gamma(\mathbf{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbf{P}^1, \mathcal{O}(N))$ is injective, so we get $n \leq N$. Since $\mathcal{O}(D) = \tilde{\varphi}^*\mathcal{O}(N)$, $d = N \deg \tilde{\varphi} = N \deg \varphi$. Therefore $3 \leq \deg \varphi = d/N \leq d/n < 3$, which is impossible.
2. If $\deg \varphi = 1$, then $d' = d$, $g' = g$ and $m = [(d'-1)/(n-1)] = [2 + (c+1)/(n-1)]$.
   (a) If $n \geq c + 3$, then $m = 2$, $\varepsilon = c + 1$ and
   $g \leq n - 1 + 2(c+1)$ (*by Castelnuovo*)
   $= 2d - 3n + 1$ (*by* $c = d - 2n$)
   $\leq d - 2$ (*by* $3n \geq d + 3$).
   This contradicts $d \leq g - 1$.
   (b) If $n = c + 2$, then $m = 3$, $\varepsilon = 0$ and $g \leq 3(n-1) = d - 1$, which also conflicts.

As a consequence we get $\deg \varphi = 2$, so $\tilde{\varphi}$ is a double covering map. Therefore $d' = d/2$; hence d and c are even. Again using Castelnuovo's lemma, we get $g' \leq d' - n = c/2$. Since $g - 1 \geq d = c + 2n \geq 3c + 4 \geq 6g' + 4$, Theorem 1 is proved. Q.E.D.

Proposition. *In the above notation, let σ be the involution of C compatible with $\tilde{\varphi}$. Then D is invariant under the action of σ^*.*

Proof. If x in the support of D is not a branch point, then $\mathrm{B_s}|D - x| = \{\sigma(x)\}$ because $\tilde{\varphi}$ is a double covering, so $\sigma(x)$ lies in the support of D. Q.E.D.

Theorem 2. *Let C (resp. C') be a curve of genus g (resp. g') and $g \geq 4g' - 2$ with a double covering $\pi : C \to C'$. Then there is an effective divisor D on C with $c(D) = 2g'$ for any even degree with $2(2g' - 2) \leq \deg D \leq 2[(g-1)/2]$.*

Proof. Suppose the ramification divisor of π on $P_1 + \cdots + P_{2(g-2g'+1)}$ on C and suppose σ is the involution of C which is compatible with π. Here the number of ramification points is calculated by the Hurwitz formula. Take z in the function field $k(C)$ of C with $\sigma^* z = -z$. Multiplying z by an element of $\pi^* k(C')$ if necessary, we may assume that $\mathrm{div}(z) = \sum_{i=1}^{2(g-2g'+1)} P_i - \pi^* Q$ where Q is some divisor on C'.

Since $g \geq 4g' - 2$, we can take an effective divisor D_0 on C' such that $(g-1)/2 \geq d_0 := \deg D_0 \geq 2g' - 2$. We set $D := \pi^* D_0$ and verify that D satisfies the desired condition. Since σ^* acts on $\Gamma(D)$, $\Gamma(D) \cong \Gamma^1 \oplus \Gamma^{-1}$ where $\Gamma^{\pm 1}$ is the eigenspace with eigenvalue ± 1 respectively. Take f in $k(C')$. Then

$$\begin{aligned}\pi^* f \cdot z \in \Gamma^{-1} &\Leftrightarrow \mathrm{div}(\pi^* f) + \textstyle\sum P_i - \pi^* Q + \pi^* D_0 \geq 0 \\ &\Leftrightarrow \mathrm{div}(f) - Q + D_0 \geq 0 \\ &\quad \text{(because the order of the pull-back of a divisor on } C' \text{ at any point of } C \text{ is even)} \\ &\Leftrightarrow f \in \Gamma(D_0 - Q).\end{aligned}$$

But since $\deg(D_0 - Q) = d_0 - (g - 2g' + 1) \leq (g-1)/2 - g + 2g' - 1 = (4g' - 3 - g)/2 < 0$, $\dim(\Gamma^{-1}) = 0$ so that $\Gamma(D) \cong \Gamma^1$. We get $l(D) = l(D_0) = 1 - g' + d_0$, $c(D) = 2d_0 - 2(1 - g' + d_0) + 2 = 2g'$. Q.E.D.

Corollary. *Let C be a non-hyperelliptic curve of genus $g \geq 13$ or $g = 11$. Then C is 1-hyperelliptic if and only if there is a divisor D with degree $g - 1$ and $l(D) = [(g-1)/2]$.*

Proof. If there is a divisor D with $\deg D = g - 1$ and $l(D) = [(g-1)/2]$, then $c(D) = 2$ or 3. Put $D' := D - \mathrm{Fix}\ (D)$; then D' is base-point-free, $c(D') \leq 2$ and $l(D') = l(D) = [(g-1)/2] \geq c(D') + 3$ because $c(D) = 2$ if $g = 11$. By Theorem 1, $c(D') = 2$ and C is 1-hyperelliptic since C is not hyperelliptic. Conversely, if C is 1-hyperelliptic then there is such a divisor by Theorem 2.

Remark. We can also prove the following statement in a way similar to the proof of the above corollary. *If $g \geq 7$, then C is hyperelliptic if and only if there is a divisor D with degree $g - 1$ and $l(D) = [(g+1)/2]$.*
This shows that the condition '$2D \sim K_C$' is not necessary in Theorem 3.1 [3].

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