Some Integral Transforms in the Space of Entire Functions of Exponential Type

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Abstract: Some integral transform with the Humbert confluent hypergeometric function of two variables Φ_1 in the kernel is proved to be an isomorphism in the space of entire functions of exponential type.

Key words: Integral transform; confluent hypergeometric function; entire function of exponential type.

1. Introduction. Let $E^\sigma (\sigma > 0)$ be the class of entire functions of type at most σ, that means $f \in E^\sigma$ if and only if $f(z) = O(e^{(\sigma+\varepsilon)|\mathrm{Im}\, z|})$ as $|z| \to \infty$ for every $\varepsilon > 0$ [1]. The intersection of the restriction of E^σ on $\boldsymbol{R}$ with $L_2(\boldsymbol{R})$ is denoted by M^σ.

It is well known (Paley-Wiener Theorem) [1] that $f \in M^\sigma$ if and only if f is the Fourier transform of a function $\tilde{f} \in L_2(\boldsymbol{R})$ with compact support from $[-\sigma, \sigma]$:

$$(1)\quad f(x) = \int_{-\sigma}^{\sigma} \tilde{f}(y) e^{ixy} dy, \quad \tilde{f}(y) \in L_2(-\sigma, \sigma).$$

The space M^σ plays an important role in the theories of distribution and partial differential equations. In this paper we establish some integral transform that is an isomorphism on M^σ. In general, classical integral transforms as well as integral transforms studied recently, e.g. Srivastava-Buschman [4], Vu Kim Tuan [5], also the table of integral transforms in Prudnikov *et al.* [3], are mostly considered in L_p and other spaces.

2. Some preliminary results. We need some elementary facts.

Lemma 1. *Let* $k \in L_1(\boldsymbol{R})$ *and* $f \in M^\sigma$. *Then the convolution*

$$(2)\quad g(x) = (k * f)(x) = \int_{-\infty}^{\infty} k(x-y) f(y) dy$$

also belongs to M^σ.

In fact, in this case we have [1]

$$(3)\quad \hat{g} = \widehat{k * f} = \hat{k} \cdot \hat{f},$$

where $\hat{f}$ is the Fourier transform of f understood either in $L_1(\boldsymbol{R})$ or $L_2(\boldsymbol{R})$-meaning [1]

$$(4)\quad \hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{ixy} dy \text{ if } f \in L_1(\boldsymbol{R}),$$

$$(5)\quad \hat{f}(x) = \lim_{N\to\infty} \int_{-N}^{N} f(y) e^{ixy} dy \text{ if } f \in L_2(\boldsymbol{R})$$

with the limit being taken in L_2-norm. For $f \in L_2(\boldsymbol{R})$ and $k \in L_1(\boldsymbol{R})$, the convolution $k * f$ belongs to $L_2(\boldsymbol{R})$ [1]. Furthermore, $\mathrm{supp}(\hat{f}) \subset [-\sigma, \sigma]$ according to the Paley-Wiener theorem, and hence

$$(6)\quad \mathrm{supp}(\widehat{k * f}) = \mathrm{supp}(\hat{k} \cdot \hat{f}) \subset \mathrm{supp}\,(\hat{f}) \subset [-\sigma, \sigma],$$

which means the support of $\widehat{k * f}$ is included in $[-\sigma, \sigma]$. The Paley-Wiener theorem implies now that $k * f \in M^\sigma$.

Lemma 2. *Let* $k \in M^\sigma$ *and let* $\hat{k}$ *and* $1/\hat{k}$ *be both bounded. Then convolution* (2) *is an isomorphism on* M^σ.

In fact, if $f \in M^\sigma$, then both f and k belong to $L_2(\boldsymbol{R})$. Therefore, formula (3) remains valid. Since $\hat{k}$ is bounded, one can follow the proof of Lemma 1 to obtain that $g \in M^\sigma$. Let now $g \in M^\sigma$. Putting

$$(7)\quad \hat{f} = \frac{1}{\hat{k}} \cdot \hat{g}.$$

Since $1/\hat{k}$ is bounded and $\hat{g} \in L_2(\boldsymbol{R})$, it follows that $\hat{f} \in L_2(\boldsymbol{R})$. Furthermore, $\mathrm{supp}\,(\hat{f}) = \mathrm{supp}\,(\hat{g}) \subset [-\sigma, \sigma]$. Hence $f \in M^\sigma$. From (7) we have that $\hat{g}$ can be decomposed in the form (3), that means g can be expressed as the convolution of k and f in the form (2), where $f \in M^\sigma$. Lemma 2 is thus proved.

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Corollary. *M^σ is the space of all square integrable eigenfunction of the operator*

$$(8)\qquad f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\sigma(x-y)}{x-y}f(y)\,dy.$$

Indeed, since the Fourier transform of $\sin\sigma y/\pi y$ is the characteristic function $\chi_{[-\sigma,\sigma]}(x)$ of the interval $[-\sigma,\sigma]$, equation (8) is equivalent to

$$(9)\qquad \hat{f}(x)=\chi_{[-\sigma,\sigma]}(x)\hat{f}(x).$$

Equation (9) has solutions if and only if $\operatorname{supp}(\hat{f})\subset[-\sigma,\sigma]$ that means $f\in M^\sigma$.

3. Some integral transforms. Since $f(x)\in M^\sigma$ if and only if $f(x/\sigma)\in M^1$, we will consider only M^1, for simplicity. Let

$$(10)\ \hat{k}(x)=\pi 2^{\beta-i\gamma}a^{-\beta}e^{b/2}\frac{\Gamma(2+i\gamma)}{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}\cdot\chi_{[-1,1]}(x)(1+x)^{i\alpha}(1-x)^{i\gamma-i\alpha}\left(\frac{2}{a}-1-x\right)^{-\beta}e^{bx/2},$$

where $\alpha,\gamma\in\boldsymbol{R}$, $a\notin[1,\infty)$. Then $\hat{k}(x)$ and

$$(11)\ \frac{1}{\hat{k}(x)}=\frac{1}{\pi}2^{i\gamma-\beta}a^{\beta}e^{-b/2}\frac{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}{\Gamma(2+i\gamma)}\cdot\chi_{[-1,1]}(x)(1+x)^{-i\alpha}(1-x)^{i\alpha-i\gamma}\left(\frac{2}{a}-1-x\right)^{\beta}e^{-bx/2}$$

are both bounded. Therefore by virtue of Lemma 2, the transform (2) with the kernel $k(x)$ is an isomorphism on M^1. We will find the form of $k(x)$ now. We have

$$k(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{k}(y)e^{-ixy}dy$$
$$=2^{\beta-i\gamma-1}a^{-\beta}e^{b/2}\frac{\Gamma(2+i\gamma)}{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}\cdot\int_{-1}^{1}(1+y)^{i\alpha}(1-y)^{i\gamma-i\alpha}\left(\frac{2}{a}-1-y\right)^{-\beta}e^{by/2-ixy}dy.$$

Putting $y=2t-1$, we obtain

$$(12)\qquad k(x)=\frac{\Gamma(2+i\gamma)}{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}\cdot e^{ix}\int_0^1 t^{i\alpha}(1-t)^{i\gamma-i\alpha}(1-at)^{-\beta}e^{bt-2ixt}dt.$$

The integral in (12) can be expressed through the Humbert confluent hypergeometric function of two variables $\Phi_1(\alpha,\beta,\gamma;x,y)$ [2]

$$(13)\qquad \Phi_1(\alpha,\beta,\gamma;x,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}e^{yt}dt$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}}\frac{x^m y^n}{m!\,n!},$$

where $(\alpha)_m=\Gamma(\alpha+m)/\Gamma(\alpha)$ is the Pochhammer symbol [2]. We get

$$(14)\quad k(x)=e^{ix}\Phi_1(1+i\alpha,\beta,2+i\gamma;a,b-2ix).$$

Similarly, from (11) we obtain

$$(15)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ixy}}{\hat{k}(y)}dy=\left|\frac{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}{\pi\Gamma(2+i\gamma)}\right|^2\cdot e^{ix}\Phi_1(1-i\alpha,-\beta,2-i\gamma;a,-b-2ix).$$

Thus we have

Theorem. *Let $\alpha,\gamma\in\boldsymbol{R}$, $a\notin[1,\infty)$. The integral transform*

$$(16)\quad g(x)=\int_{-\infty}^{\infty}e^{i(x-y)}\Phi_1(1+i\alpha,\beta,2+i\gamma;a,b+2i(y-x))f(y)\,dy$$

is an isomorphism on M^1 and the inverse transform has the form

$$(17)\quad f(x)=\left|\frac{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}{\pi\Gamma(2+i\gamma)}\right|^2\cdot\int_{-\infty}^{\infty}e^{i(x-y)}\Phi_1(1-i\alpha,-\beta,2-i\gamma;a,-b+2i(y-x))g(y)\,dy.$$

If, moreover, $a\in(-\infty,1)$ and $\operatorname{Re}\beta=\operatorname{Re}b=0$, then

$$(18)\qquad \|f\|_2=\left|\frac{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}{\pi\Gamma(2+i\gamma)}\right|\|g\|_2.$$

4. Special cases. 1) Let in (16) and (17) $\beta=b=0$. Then we obtain a pair of transforms in M^1

$$(19)\qquad g(x)=\int_{-\infty}^{\infty}e^{i(x-y)}{}_1F_1(1+i\alpha,2+i\gamma;2i(y-x))f(y)\,dy,$$

$$(20)\quad f(x)=\left|\frac{\Gamma(1+i\alpha)\Gamma(1+i\gamma-i\alpha)}{\pi\Gamma(2+i\gamma)}\right|^2\cdot\int_{-\infty}^{\infty}e^{i(x-y)}{}_1F_1(1-i\alpha,2-i\gamma;2i(y-x))g(y)\,dy,$$

where ${}_1F_1(\alpha,\gamma;x)$ is the confluent hypergeometric function [2]:

$$(21)\qquad {}_1F_1(\alpha,\gamma;x)=\sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\gamma)_m}\frac{x^m}{m!}.$$

2) Let $\gamma=\alpha$ in (19) and (20). We get

$$(22)\qquad g(x)=\int_{-\infty}^{\infty}(x-y)^{-1-i\alpha}e^{i(x-y)}\cdot\gamma(1+i\alpha,2i(x-y))f(y)\,dy,$$

$$(23)\qquad f(x)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}(x-y)^{-1+i\alpha}e^{i(x-y)}\cdot\gamma(1-i\alpha,2i(x-y))g(y)\,dy,$$

where $\gamma(\alpha,x)$ is the incomplete Gamma function [2].

3) Let in (19) and (20) $\gamma=2\alpha$. Then

$$(24)\ g(x)=\int_{-\infty}^{\infty}(y-x)^{-i\alpha-1/2}J_{1/2+i\alpha}(y-x)f(y)\,dy,$$

$$(25)\qquad f(x)=\frac{\alpha}{8\sinh\pi\alpha}\int_{-\infty}^{\infty}(y-x)^{i\alpha-1/2}$$

$$\cdot J_{1/2-i\alpha}(y-x)g(y)dy,$$

where $J_\nu(x)$ is the Bessel function of the first kind [2].

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