On the Structure of Painlevé Transcendents with a Large Parameter. II.

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§0. Introduction. The purpose of this note is to report a result on the structure of 2-parameter formal solutions of the Painlevé equations with a large parameter η, which are tabulated in Table 0.1 below. The formal solutions to be considered here have been constructed in [1] by the so-called multiple-scale analysis, and the main result (Theorem 2.1) of this note asserts that any of them can be locally reduced to a 2-parameter formal solution of the first Painlevé equation (P_{I}); this is a natural generalization of the result on 0-parameter solutions reported in our precedent note [3]. (See [4] for the details of the proof of the results announced in [3].)

Table 0.1. Painlevé equations with a large parameter η.

(P_{I}) $$\frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

(P_{II}) $$\frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \alpha).$$

(P_{III}) $$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2\alpha_\infty\lambda^3 + \frac{\alpha'_\infty}{t}\lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda}\right].$$

(P_{IV}) $$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + (t^2 + 4\alpha_1)\lambda - \frac{4\alpha_0}{\lambda}\right].$$

(P_{V}) $$\frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right) + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(\alpha_0 + \alpha_\infty) - \alpha_0\frac{1}{\lambda^2} - \alpha_2\frac{t}{(\lambda-1)^2} - \alpha_1 t^2\frac{\lambda+1}{(\lambda-1)^3}\right].$$

(P_{VI}) $$\frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2} + \eta^2\left\{(\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty) - \alpha_0\frac{t}{\lambda^2} + \alpha_1\frac{t-1}{(\lambda-1)^2} - \alpha_t\frac{t(t-1)}{(\lambda-t)^2}\right\}\right].$$

The details of this note shall be published elsewhere. We sincerely thank Professor T. Aoki for the stimulating discussions with him on the subjects discussed here.

§1. A canonical Schrödinger equation (Can) near the double turning point and its isomonodromic deformation. In this note we use the same notions and notations as in [3] except that the formal solution $\lambda_J (J = \mathrm{I}, \mathrm{II}, \cdots \mathrm{VI})$ of (P_J) considered in [3] and [4] is denoted by $\lambda_J^{(0)}$ here; in particular (SL_J) denotes the Schrödinger equation tabulated in Table 1.2 of [4], K_J denotes the Hamiltonian tabulated in Table 1.3 of [4], and $S_{J,\mathrm{odd}}$ denotes the odd part of a solution S_J of the Riccati equation

(1.1) $$S_J^2 + \frac{\partial S_J}{\partial x} = \eta^2 Q_J$$

associated with (SL_J). (Cf. [1], Definition 2.1.) In order to save space we also refer the reader to [4] for the definition of the coefficient A_J of the deformation equation (D_J) for (SL_J), i.e.,

(D_J) $$\frac{\partial \psi}{\partial t} = A_J\frac{\partial \psi}{\partial t} - \frac{1}{2}\frac{\partial A_J}{\partial x}\psi.$$

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We only note that A_J contains the factor $(x-\lambda_J)^{-1}$, e.g.,

$$A_J=\frac{1}{2(x-\lambda_J)}\ (J=\mathrm{I},\ \mathrm{II}),$$

$$A_{\mathrm{VI}}=\frac{(\lambda_{\mathrm{VI}}-t)x(x-1)}{t(t-1)(x-\lambda_{\mathrm{VI}})},\quad etc.$$

In what follows we substitute into (λ, ν) in the coefficients of the potential Q_J the 2-parameter solution (λ_J, ν_J) of the Hamiltonian system (H_J):

$$(1.2)_J\qquad \begin{cases}\dfrac{d\lambda}{dt}=\eta\dfrac{\partial K_J}{\partial\nu}\\ \dfrac{d\nu}{dt}=-\eta\dfrac{\partial K_J}{\partial\lambda}.\end{cases}$$

Then $\tilde{x}=\lambda_{J,0}(t)$ is a double turning point of (SL_J), and we can find a WKB-theoretic formal transformation

$$(1.3)\quad x=x(\tilde{x},t,\eta)=\sum_{\substack{j\geq 0\\ j\neq 1}}x_{j/2}(\tilde{x},t,\eta)\eta^{-j/2}$$

near the double turning point $\lambda_{J,0}(t)$ (for generic t) so that (SL_J) may be brought into the following canonical Schrödinger equation (Can). (See Theorem 3.1 of [1] for the precise statement.)

$$(Can)\quad \left(-\frac{\partial^2}{\partial x^2}+\eta^2Q_{can}(x,t,\eta)\right)\psi=0$$

with

$$(1.4)\qquad Q_{can}=4x^2+\eta^{-1}E(t,\eta)+\frac{\eta^{-3/2}\rho(t,\eta)}{x-\eta^{-1/2}\sigma(t,\eta)}+\frac{3\eta^{-2}}{4(x-\eta^{-1/2}\sigma(t,\eta))^2}$$

where

$$(1.5)\qquad E=\rho^2-4\sigma^2.$$

Here the parameters σ and ρ are related to (λ_J, ν_J) in the following manner:

$$(1.6)\qquad \sigma=\eta^{1/2}x(\lambda_J(t,\eta),t,\eta)$$

and

$$(1.7)\quad \rho=-\frac{\eta^{1/2}\nu_J}{\dfrac{\partial x}{\partial\tilde{x}}(\lambda_J,t,\eta)}-\frac{3}{4}\eta^{-1/2}\frac{\dfrac{\partial^2x}{\partial\tilde{x}^2}(\lambda_J,t,\eta)}{\left(\dfrac{\partial x}{\partial\tilde{x}}(\lambda_J,t,\eta)\right)^2}.$$

We now try to isomonodromically deform (Can) (in the sense of [2]), forgetting the origin (i.e., relations (1.6) and (1.7)) of the parameters ρ and σ at the moment.

Proposition 1.1. *Let A_{can} denote*

$$(1.8)\qquad \frac{1}{2(x-\eta^{-1/2}\sigma(t,\eta))}.$$

Then the following equation

$$(D_{can})\qquad \frac{\partial\psi}{\partial t}=A_{can}\frac{\partial\psi}{\partial x}-\frac{1}{2}\frac{\partial A_{can}}{\partial x}\psi$$

is in involution with (Can) if ρ and σ satisfies the following Hamiltonian system:

$$(H_{can})\qquad \begin{cases}\dfrac{d\rho}{dt}=-4\eta\sigma\\ \dfrac{d\sigma}{dt}=-\eta\rho\end{cases}.$$

Although the proof of this proposition is a straightforward one, the result plays an important role in our reasoning given below; as a solution $(\rho_{can}, \sigma_{can})$ of (H_{can}) can be readily written down explicitly as a sum of exponential functions, we can choose a formal transformation $t(\tilde{t},\eta)$ using the structure of $(\rho_{can}, \sigma_{can})$ so that the transformation together with the transformation $x(\tilde{x},\tilde{t},\eta)$ given by (1.3) may bring (SL_J) and (D_J) simultaneously into (Can) and (D_{can}). To be more precise, we find Proposition 1.2 below by the aid of the following Lemma 1.1 and Lemma 1.2. Until the end of this section the symbols t, λ_J and ν_J in Q_J shall be respectively replaced by $\tilde{t}$, $\tilde{\lambda}_J$ and $\tilde{\nu}_J$. For the sake of clarity of notations we also use symbols $\tilde{\sigma}_J$ and $\tilde{\rho}_J$ to denote the functions σ and ρ given respectively by (1.6) and (1.7) through the transformation $x(\tilde{x},\tilde{t},\eta)$. In accordance with this convention we use symbols E_{can} and $\tilde{E}_J$ to denote $\rho_{can}^2-4\sigma_{can}^2$ and $\tilde{\rho}_J^2-4\tilde{\sigma}_J^2$ respectively. In what follows we fix an open neighborhood $\tilde{V}$ of a fixed generic point $\tilde{t}_*$ in a Stokes curve for $\tilde{\lambda}_J^{(0)}$ emanating from a turning point $\tilde{r}$ for $\tilde{\lambda}_J^{(0)}$.

Lemma 1.1. *The series E_{can} and $\tilde{E}_J$ are independent of t and $\tilde{t}$ respectively.*

Lemma 1.2. *There exists a formal series $t(\tilde{t},\eta)=\sum_{j\geq 0}t_{j/2}(\tilde{t},\eta)\eta^{-j/2}$ so that the following conditions may be satisfied:*

(1.9) *$t_{j/2}(\tilde{t},\eta)$ is holomorphic on $\tilde{V}$,*

(1.10) *$\rho_{can}(t(\tilde{t},\eta),\eta)=\tilde{\rho}_J(\tilde{t})$ and $\sigma_{can}(t(\tilde{t},\eta),\eta)=\tilde{\sigma}_J(\tilde{t})$ hold,*

(1.11) *$t_0(\tilde{t},\eta)=\tilde{\phi}_J(\tilde{t})/2$ holds, where $\tilde{\phi}_J(\tilde{t})$ denotes the integral*

$$\int_{\tilde{r}}^{\tilde{t}}\sqrt{\frac{\partial\tilde{F}_J}{\partial\tilde{\lambda}}(\tilde{\lambda}_{J,0}(s),s)}\,ds\ \text{(cf. [4], §2),}$$

(1.12) *$t_{1/2}(\tilde{t},\eta)$ identically vanishes,*

(1.13) *$t_{j/2}(j\geq 2)$ has the following form:*

$$\sum_{k=0}^{j-2}s_{j-2-2k}(\tilde{t})e^{(j-2-2k)\tilde{\phi}_J(\tilde{t})\eta}.$$

The independency of E_{can} on t is an immediate consequence of the definition of E_{can} and the explicit form of (H_{can}), while the independency of $\tilde{E}_J$ on $\tilde{t}$ is based upon the following properties (cf. [1], §2 and §3):

$$\frac{\partial}{\partial \tilde{t}} \tilde{S}_{J,\mathrm{odd}} = \frac{\partial}{\partial \tilde{x}} (\tilde{A}_J \tilde{S}_{J,\mathrm{odd}}), \tag{1.14}$$

$$\oint_{\text{around } \tilde{x}=\tilde{\lambda}_{J,0}(\tilde{t})} \tilde{S}_{J,\mathrm{odd}} d\tilde{x} = \frac{\pi i}{2} \tilde{E}_J. \tag{1.15}$$

Thanks to Lemma 1.1, we can readily require

$$E_{can} = \tilde{E}_J\,; \tag{1.16}$$

this is just a relation between the parameters contained in $(\rho_{can}, \sigma_{can})$ and those in $(\tilde{\rho}_J, \tilde{\sigma}_J)$. On the other hand, the proof of Lemma 1.2 is given by the induction on j that makes full use of Lemma 1.1 and (1.16). We note that in the course of the argument $t_{j/2}(j \geq 2$, even integer) is determined modulo an additive constant, which shall be fixed later. (Cf. §2.)

Proposition 1.2. *Let $\psi(x, t, \eta)$ be a WKB solution of (Can) that satisfies (D_{can}) also, and let $\tilde{\psi}(\tilde{x}, \tilde{t}, \eta)$ denote the following function determined by the transformation $x(\tilde{x}, \tilde{t}, \eta)$ given by* (1.3) *together with the transformation $t(\tilde{t}, \eta)$ given in Lemma* 1.2:

$$\tilde{\psi}(\tilde{x}, \tilde{t}, \eta) = \left(\frac{\partial x(\tilde{x}, \tilde{t}, \eta)}{\partial \tilde{x}}\right)^{-1/2} \psi(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta). \tag{1.17}$$

Then $\tilde{\psi}$ satisfies both (SL_J) and (D_J) near the double turning point.

The proof of this proposition is attained by verifying

$$\tilde{A}_J \frac{\partial x}{\partial \tilde{x}} - \frac{\partial x}{\partial \tilde{t}} - A_{can} \frac{\partial t}{\partial \tilde{t}} = 0\,; \tag{1.18}$$

as is shown in the proof of Proposition2.2 of [4], (1.18) guarantees that $\tilde{\psi}$ satisfies not only (SL_J) but also (D_J).

§2. Local equivalence of 2-parameter Painlevé transcendents. The purpose of this section is to state our main result (Theorem 2.1) to the effect that any 2-parameter formal solution of (P_J) $(J =$ II, III, . . . , VI) constructed in §1 of [1] can be transformed into a 2-parameter formal solution of (P_{I}). The transformation is found, as in the case of 0-parameter solutions, through the transformation of (SL_J) into (SL_{I}). As the analytic structure of WKB solutions of (SL_J) with 2-parameter solutions of (H_J) in its coefficients behaves much wilder than that of WKB solutions of (SL_J) with 0-parameter solutions in its coefficients, a straightforward generalization of the argument given in [4] seems to be formidably difficult; we circumvent the trouble by making explicit use of (Can) and (D_{can}). In what follows, we put $\sim$ to variables and functions relevant to (SL_J). We also use the symbol $(x_{\mathrm{I}}(x, t, \eta), t_{\mathrm{I}}(t, \eta))$ (resp., $(x_J(\tilde{x}, \tilde{t}, \eta), t_J(\tilde{t}, \eta))$) to denote the transformation discussed in §1 that brings (SL_{I}) and (D_{I}) (resp., (SL_J) and (D_J)) into (Can) and (D_{can}) near the double turning point.

Before stating our main result let us recall some geometric facts relating the Stokes geometry of (SL_J) and that for $\tilde{\lambda}_J^{(0)}$. (See §2 of [4] for the details.) Let $\tilde{t}_*$ be a point in a Stokes curve for $\tilde{\lambda}_J^{(0)}$ emanating from a simple turning point $\tilde{r}$ for $\tilde{\lambda}_J^{(0)}$. Then, unless $\tilde{t}_* = \tilde{r}$, there exist a simple turning point $\tilde{a}(\tilde{t})$ and a Stokes curve $\tilde{\gamma}$ of (SL_J) such that $\tilde{\gamma}$ joins $\tilde{a}(\tilde{t})$ and the double turning point $\tilde{\lambda}_{J,0}(\tilde{t})$. The core of our argument is the construction of a transformation that brings (SL_J) into (SL_{I}) on a neighborhood of $\tilde{\gamma}$, and in stating our main result (Theorem 2.1 below), we consider the problem in this geometric setting.

Theorem 2.1. *For each 2-parameter formal solution $(\tilde{\lambda}_J, \tilde{\nu}_J)$ of (H_J) there exists a 2-parameter formal solution $(\lambda_{\mathrm{I}}, \nu_{\mathrm{I}})$ of (H_{I}) for which the following holds:*

There exist a neighborhood $\tilde{U}$ of $\tilde{\gamma}$, a neighborhood $\tilde{V}$ of $\tilde{t}_*$ and holomorphic functions $x_{j/2}(\tilde{x}, \tilde{t}, \eta)$ $(j = 0, 1, 2, \ldots)$ on $\tilde{U} \times \tilde{V}$ and $t_{j/2}(\tilde{t}, \eta)$ on $\tilde{V}$ which satisfy the following relations:

(i) *The function t_0 is independent of η and satisfies*

$$\tilde{\phi}_J(\tilde{t}) = \phi_{\mathrm{I}}(t_0(\tilde{t})), \tag{2.1}$$

(ii) *The function x_0 is also independent of η and satisfies $x_0(\tilde{\lambda}_{J,0}(\tilde{t}), \tilde{t}) = \lambda_{\mathrm{I},0}(t_0(\tilde{t}))$ and $x_0(\tilde{a}(\tilde{t}), \tilde{t}) = -2\lambda_{\mathrm{I},0}(t_0(\tilde{t}))(= a(t_0(\tilde{t})))$,*

(iii) *$\partial x_0 / \partial \tilde{x}$ never vanishes on $\tilde{U} \times \tilde{V}$,*

(iv) *$x_{1/2}$ and $t_{1/2}$ vanish identically,*

(v) *For $x(\tilde{x}, \tilde{t}, \eta) = \sum_{j \geq 0} x_{j/2} \eta^{-j/2}$ and $t(\tilde{t}, \eta) = \sum_{j \geq 0} t_{j/2} \eta^{-j/2}$, the following relations hold:*

$$x(\tilde{\lambda}_J(\tilde{t}, \eta), \tilde{t}, \eta) = \lambda_{\mathrm{I}}(t(\tilde{t}, \eta), \eta), \tag{2.2}$$

$$\tilde{Q}_J(\tilde{x}, \tilde{t}, \eta) = \left(\frac{\partial x(\tilde{x}, \tilde{t}, \eta)}{\partial \tilde{x}}\right)^2 Q_{\mathrm{I}}(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta) - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \tilde{t}, \eta)\,; \tilde{x}\}, \tag{2.3}$$

where the 2-parameter solutions in question of (H_J) and (H_{I}) are substituted into (λ, ν) in the coefficients of $\tilde{Q}_J$ and Q_I respectively, and $\{x\,; \tilde{x}\}$ denotes the Schwarzian derivative.

Note that, among others, the relation (2.2) describes the local equivalence of the 2-parameter formal solutions $\tilde{\lambda}_J$ and λ_{I} of (P_J) and (P_{I}).

Our strategy of the proof of Theorem 2.1 is as follows:

Near the double turning point we can choose $t_{\mathrm{I}}^{-1}(t_J(\tilde{t}, \eta), \eta)$ and $x_{\mathrm{I}}^{-1}(x_J(\tilde{x}, \tilde{t}, \eta), t_J(\tilde{t}, \eta), \eta)$ as $t(\tilde{t}, \eta)$ and $x(\tilde{x}, \tilde{t}, \eta)$ so that they satisfy (2.3). We cannot, however, expect $x(\tilde{x}, \tilde{t}, \eta)$ thus defined can be extended over a neighborhood of $\{\tilde{a}(\tilde{t})\} \times \tilde{V}$; the free constant remaining in the definition of $t_{\mathrm{I},j/2}(t)$ should be suitably adjusted. To find the correct $t(\tilde{t}, \eta)$, we consider

(2.4) $$y(\tilde{x}, \tilde{t}, \eta) = \sum_{j\geq 0} y_{j/2}(\tilde{x}, \tilde{t}, \eta)\eta^{-j/2}$$

which satisfies

(2.5) $$\tilde{S}_{J,\mathrm{odd}}(\tilde{x}, \tilde{t}, \eta) = \frac{\partial y}{\partial \tilde{x}} S_{\mathrm{I}}(y(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta)$$

near the simple turning point $\tilde{x} = \tilde{a}(\tilde{t})$, and seek for the condition that makes x to coincide with y. Note that (2.5) is another way of expressing the condition (2.3) stated in terms of the potential; (2.5) is more convenient in our discussion (e.g., in showing the regular singular character of the differential equation for x near the double turning point). A crucial point in our reasoning is to consider

(2.6) $$R(x, t, \eta) = \int_{-2\lambda_{\mathrm{I},0}(t)}^{x} \eta^{-1} S_{\mathrm{I},\mathrm{odd}}(z, t, \eta)\, dz.$$

Making full use of deformation equations, we can verify

(2.7) $$R(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta) - R(y(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta) = \sum_{j\geq 0} C_{j/2}\eta^{-j/2}$$

holds for some constant $C_{j/2}$ which is independent both of $\tilde{x}$ and $\tilde{t}$. We then use an induction on j to show the following:

$(\mathscr{C})_j$ A correct choice of $t_{j/2}$ entails the vanishing of $C_{j/2}$ and the coincidence of $x_{j/2}$ and $y_{j/2}$.

We note that $C_{n/2}$ automatically vanishes for an odd integer n, reflecting the instanton structure of relevant quantities.

Once we establish $(\mathscr{C})_j$ for any j, then we obtain the transformation $x(\tilde{x}, \tilde{t}, \eta)$ and $t(\tilde{t}, \eta)$ which satisfy (2.3) and whose coefficients are holomorphic on $\tilde{U} \times \tilde{V}$ and on $\tilde{V}$ respectively. The proof of (2.2) can be readily given also.

Remark 2.1. The equation (2.2) implies the relation between the parameters contained in λ_{I} and those in $\tilde{\lambda}_J$. See §4 of [1] for some explicit computation in the case of $J = \mathrm{II}$. The relation should be important in our future understanding of the connection formula for the Painlevé transcendents (cf. [5]).

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