Euler's Discretization Revisited

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1. Introduction. It is well known that there are some kinds of scalar ordinary differential equations of which corresponding difference equations (Euler's discretization) are chaotic in the sense of Li-Yorke [1]. This work was done by Yamaguti-Matano in [2]. In this paper, it will be shown that the globally asymptotically stable differential equation which has only one stable equilibrium point is apt to turn into chaos in the corresponding difference equation under some conditions. Moreover, we will give an example such that for any small Δt the difference equation is chaotic.

2. Euler's discretization and Yamaguti–Matano theorem. For the scalar ordinary differential equation

$$\frac{du}{dt} = f(u), \tag{1}$$

Euler's discretization of (1) is defined as follows:

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n), \text{ so } x_{n+1} = x_n + \Delta t f(x_n).$$

Let Euler's difference equation $F_{\Delta t}(x)$ be

$$F_{\Delta t}(x) = x + \Delta t f(x), \tag{2}$$

then the next theorem is obtained.

Theorem (Yamaguti-Matano). Assume $f(u)$ holds condition $(*)$.

$$(*)\begin{cases} (i)\, f(u) \text{ is continuous in } R^1 \\ (ii)\, f(0) = f(\bar{u}) = 0\ (\exists\, \bar{u} > 0) \\ (iii)\, f(u) > 0 \qquad (0 < \forall\, u < \bar{u}) \\ (iv)\, f(u) < 0 \qquad (\bar{u} < \forall\, u < K) \\ \qquad\qquad K \text{ is a constant } (\leq +\infty) \end{cases}$$

Then,

(i) there exists a positive constant c_1 such that for any $\Delta t > c_1$ the difference equation (2) is chaotic in the sense of Li-Yorke.

(ii) Suppose in addition that $K = +\infty$; then there exists another constant c_2, $0 < c_1 < c_2$, such that for any $0 \leq \Delta t \leq c_2$ the map $F_{\Delta t}$ has an invariant finite interval $[0, \alpha_{\Delta t}]$ (i. e., $F_{\Delta t}$ maps $[0, \alpha_{\Delta t}]$ into itself) with $\alpha_{\Delta t} > \bar{u}$. Moreover, when $c_1 < \Delta t \leq c_2$, the above-mentioned chaotic phenomenon occurs in this invariant interval.

In this theorem, (1) have two equilibrium points and chaotic phenomenon occurs around the stable equilibrium point $\bar{u}$. But there are the differential equations with only one stable equilibrium point which turn into chaos. Now we consider three cases as follows: for $u < 0$, $f(u)$ is

$$\begin{cases} \text{Type } A;\ \text{bounded} \\ \text{Type } B;\ O((-u)^{\alpha}) \ (u \to -\infty) \ (0 < \exists\, \alpha < 1) \\ \text{Type } C;\ O((-u)^{1}) \ (u \to -\infty). \end{cases}$$

It will be shown that for each case chaos occurs under some conditions.

3. Type A. From now on, assume $f(u)$ holds following conditions $(**)$.

$$(**)\begin{cases} (i)\, f(u) \text{ is continuous in } R^1 \\ (ii)\, f(0) = 0 \\ (iii)\, f(u) > 0 \ (\forall\, u < 0) \\ (iv)\, f(u) < 0 \ (\forall\, u > 0) \end{cases}$$

Theorem A. Assume $f(u)$ holds $(**)$ and the next conditions.

$$\begin{cases} (v)\, f(u) \leq M \qquad (\forall\, u < 0) \\ \qquad\qquad M \text{ is a constant} (< +\infty) \\ (vi)\, f(c_0) = -2M \ (\exists\, c_0 > 0) \end{cases}$$

Then there exists a positive T such that for any $\Delta t \geq T$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Theorem A. To prove chaos in the sense of Li-Yorke, it is enough to show the existence of an a such that $b = F_{\Delta t}(a)$, $c = F_{\Delta t}(b)$ and $d = F_{\Delta t}(c)$ satisfy $d \leq a < b < c$. Set

$$T = \min_{x<0} \frac{c_0 - x}{f(x)} \left(\geq \frac{c_0}{M}\right).$$

$(c_0 - x)/f(x)$ is continuous and positive in $x < 0$, and also bounded from below because $\lim_{x\to-0} \frac{c_0 - x}{f(x)} = +\infty$, $\lim_{x\to-\infty} \frac{c_0 - x}{f(x)} = +\infty$. Hence T exists. And for any $\Delta t \geq T$ there is a b (< 0) such that $(c_0 - b)/f(b) = \Delta t$ holds. For c and d, $c = F_{\Delta t}(b) = b + \Delta t f(b) = c_0\ (> 0)$, $d = F_{\Delta t}(c) = c + \Delta t f(c) = c_0 - 2M\Delta t$. On the other hand, since $F_{\Delta t}(b) = b + \Delta t f(b) > b$ and $F_{\Delta t}(b - \Delta t M) = b + \Delta t(f(b - \Delta t M) - M) \leq b$, there exists an a which satisfies $F_{\Delta t}(a) = b$ in

$b - \Delta tM \leq a < b$. Compared a with d,

$$a - d \geq (b - \Delta tM) - (c_0 - 2M\Delta t) = \Delta t(M - f(b)) \geq 0.$$

Therefore $d \leq a < b < c$. An invariant finite interval is defined as follows:

$$K_1 = \max_{x \leq 0} F_{\Delta t}(x)\ (\geq c)\,(K_1 < +\infty),$$

$$K_2 = \min_{0 \leq x \leq K_1} F_{\Delta t}(x)\ (\leq d)\,(K_2 > -\infty).$$

Then $F_{\Delta t}$ maps onto $[K_2, K_1]$. Q. E. D.

Example 1 (Hata [3]). $f(u) = 1 - e^u$ satisfies the condition of Theorem A ($M = 1$, and if we take $b = -\log \Delta t$, T is given more precisely as the biggest positive solution of $e^{T-1} - 3T + 1 = 0$. $T \approx 3.1254$).

Corollary A. Assume $f(u)$ holds ($**$) and the next conditions.

$$\begin{cases} (v)\ \limsup_{u \to -\infty} f(u) = m \\ \qquad\qquad m \text{ is a constant}(< +\infty) \\ (vi)\ f(c_0) < -2m \quad (\exists\, c_0 > 0) \end{cases}$$

Then there exists a positive T such that for any $\Delta t > T$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Corollary A. Let $\varepsilon = -(2m + f(c_0))/2$.

There exists $K < 0$ such that $f(x) < m + \varepsilon (\forall x < K)$, and set T

$$T = \inf_{x<K} \frac{c_0 - x}{f(x)} \left(\geq \frac{c_0 - K}{m + \varepsilon}\right).$$

Then $\lim_{x \to -\infty} ((c_0 - x)/f(x)) = +\infty$, so for any $\Delta t > T$, there exists $b(< K)$ such that $c_0 = b + \Delta t f(b)$. For c and d, $c = F_{\Delta t}(b) = c_0$, $d = F_{\Delta t}(c) = c_0 + \Delta t f(c_0)$. The facts that $F_{\Delta t}(b) = b + \Delta t f(b) > b$ and $F_{\Delta t}(b - \Delta t(m + \varepsilon)) = b - \Delta t(m + \varepsilon) + \Delta t f(b - \Delta t(m + \varepsilon)) < b$ imply that there exists an a which satisfies $F_{\Delta t}(a) = b$ in $b - \Delta t(m + \varepsilon) < a < b$.

$$a - d > (b - \Delta t(m + \varepsilon)) - (c_0 + \Delta t f(c_0)) = \Delta t(m + \varepsilon - f(b)) > 0.$$

An invariant finite interval is the same as Theorem A. Q. E. D.

Remark 1. The conditions that

$$\lim_{u \to -\infty} f(u) = 0$$

and $f(u_0) < 0$ ($\exists\, u_0 > 0$) imply chaos. This is an extension of Yamaguti-Matano Theorem.

4. Type B. Even in the case $f(u)$ is not bounded in $u < 0$, chaos may occur. The bigger Δt becomes, the smaller both a and d become. So it is enough to check the speeds of both.

Theorem B. Assume $f(u)$ holds ($**$) and the next conditions.

$$\begin{cases} (v)\ \lim_{u \to -\infty} f(u) = O((-u)^{\alpha}) \quad (0 < \alpha < 1) \\ (vi)\ \lim_{u \to +\infty} f(u) = -O(u^{\beta}) \left(\dfrac{1}{1-\alpha} < 1 + \beta\right) \end{cases}$$

Then there exists a positive T such that for any $\Delta t > T$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Theorem B. Choose any $b < 0$ and fix. Take Δt such that $\Delta t > -b/f(b)$. $c = b + \Delta t f(b)$ (> 0), and put

$$d = c + \Delta t f(c) = b + \Delta t(f(b) + f(b + \Delta t f(b))) = -O(\Delta t^{\beta+1}).$$

$\lim_{x \to -\infty} F_{\Delta t}(x) = x^1 + O((-x)^{\alpha}) = -\infty$, so for any b and Δt, there exists an $a(< b)$ such that $a + \Delta t f(a) = b$. $\lim_{\Delta t \to +\infty} a = -\infty$, hence we can try to write $-a = O(\Delta t^m)$ and $f(a) = O((-a)^{\alpha}) = O(\Delta t^{m\alpha})$.

The orders of both sides of the equation $\Delta t f(a) = b - a$ are equal, that is

$$m\alpha + 1 = m,\ m = \frac{1}{1-\alpha},\ \text{so } a = -O(\Delta t^{1/(1-\alpha)}).$$

If $\beta + 1 < 1/(1-\alpha)$, there exists Δt_b such that $d \leq a$ for any $\Delta t > \Delta t_b$. Set $T = \inf_{b<0} \Delta t_b$. An invariant finite interval is the same as Theorem A. Q. E. D.

Corollary B. $f(u)$ is as follows:

$$f(u) = \begin{cases} (-u)^{\alpha} & (u < 0) \quad (0 < \alpha < 1) \\ -u^{\beta} & (u \geq 0). \end{cases}$$

If $\alpha < \beta$, there exists a positive T such that for any $\Delta t \geq T$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Corollary B. Take b such that $F_{\Delta t}(x)$ is maximum at b. $F_{\Delta t}(x) = x + \Delta t(-x)^{\alpha}$, and $F'_{\Delta t}(x) = 1 - \Delta t\alpha(-x)^{\alpha-1}$. From $F'_{\Delta t}(b) = 0$, $b = -(\alpha\Delta t)^{\frac{1}{1-\alpha}}$. $F_{\Delta t}(x)$ is monotone increasing in $x < b$, and also $\lim_{x \to -\infty} F_{\Delta t}(x) = -\infty$, so an a exists uniquely. This a is calculated precisely as follows: let N be a positive solution of $N = N^{\alpha} + \alpha^{\frac{1}{1-\alpha}}$. N exists uniquely and using this N we can write $a = -N\Delta t^{\frac{1}{1-\alpha}}$. For c, $c = F_{\Delta t}(b) = (\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}})\Delta t^{\frac{1}{1-\alpha}}$. Set $A = \alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}$, then $0 < A < 1$, hence $c > 0$. For d, $d = F_{\Delta t}(c) = A\Delta t^{\frac{1}{1-\alpha}} - A^{\beta}\Delta t^{(1+\frac{\beta}{1-\alpha})}$. Compared the order of a with that of d, if $\dfrac{1}{1-\alpha} < 1 + \dfrac{\beta}{1-\alpha}$ i.e. $\alpha < \beta$, there exists $T > 0$ such that for any $\Delta t \geq T$, $d \leq a$ holds. An invariant finite interval is the same as Theorem A. Q. E. D.

5. Type C. In the case of $\lim_{u \to -\infty} f(u) =$

$O((-u)^1)$, chaos may occur.

Theorem C. Assume $f(u)$ is a piecewise linear function as follows:

$$f(u) = \begin{cases} -\alpha(u-b) - b & (u < b) \\ -\beta u & (b \geq u) \quad (\alpha > 0,\ \beta > 0,\ b < 0). \end{cases}$$

If $\alpha/\beta < (2-\sqrt{3})/2$, there exists an interval $J = [\Delta t_1, \Delta t_2]$ $(0 < \Delta t_1 < \Delta t_2)$ such that for any $\Delta t \in J$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Theorem C. By scaling Δt, we can simplify $f(x)$ as follows without loss of generality. (m corresponds with α/β.)

$$f(u) = \begin{cases} -m(u-b) - b & (u < b) \quad \left(0 < m < \dfrac{2-\sqrt{3}}{2}\right) \\ -u & (b \leq u) \end{cases}$$

$$F_{\Delta t}(x) = x + \Delta t f(x) = \begin{cases} (1-m\Delta t)x + b(m-1)\Delta t & (x < b) \\ (1-\Delta t)x & (b \leq x) \end{cases}$$

$c = F_{\Delta t}(b) = (1-\Delta t)b$, $d = F_{\Delta t}(c) = (1-\Delta t)^2 b$. From $d < b < c$, $2 < \Delta t$. Also from $b = F_{\Delta t}(a) = (1-m\Delta t)a + b(m-1)\Delta t$, to exist $a(< b)$ implies $1 - m\Delta t > 0$. Hence $2 < \Delta t < 1/m$. Then,

$$a = \left(1 + \frac{\Delta t}{1-m\Delta t}\right)b,\ a - d = \frac{-b\Delta t}{1-m\Delta t}(-m\Delta t^2 + (2m+1)\Delta t - 3).$$

If $0 < m \leq (2-\sqrt{3})/2$, $-m\Delta t^2 + (2m+1)\Delta t - 3$ is positive for some Δt. Put $\Delta t_1 = \dfrac{1}{2m}(2m + 1 - \sqrt{4m^2 - 8m + 1})$, $\Delta t_2 = \dfrac{1}{2m}(2m + 1 + \sqrt{4m^2 - 8m + 1})$, then $2 < \Delta t_1 < \Delta t_2 < 1/m$, and for any $\Delta t_1 < \Delta t < \Delta t_2$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke. An invariant finite interval is the same as Theorem A. Q. E. D.

Corollary C. Assume $f(u)$ is upwards convex, monotone decreasing, C^1 class and $f(0) = 0$. If there exists $u_0 < 0$ such that $f(u_0) > (4 + 2\sqrt{3})u_0 f'(u_0)$, then there exists an interval $J = [\Delta t_1, \Delta t_2]$ $(0 < \Delta t_1 < \Delta t_2)$ such that for any $\Delta t \in J$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Corollary C. Define $m = u_0 f'(u_0)/f(u_0)$, then $0 < m < (2-\sqrt{3})/2$. And we consider $g(u)$ such that

$$g(u) = \begin{cases} -m(u-u_0) - u_0 & (u \leq u_0) \\ -u & (u_0 \leq u). \end{cases}$$

From Theorem C, the Euler's difference equation of $g(u)$ is chaotic. On the other hand, note that

$$\frac{f(u_0)}{-u_0} g(u) \geq f(u) \quad (\forall u \in (-\infty, u_0] \cup [0, +\infty)),$$

so the Euler's difference equation of $f(u)$ is also chaotic. Q. E. D.

Example 2.

$$f(u) = \begin{cases} -u + 7 & (u < -1) \\ -7u^2 - 15u & (-1 \leq u) \end{cases}$$

Example 3. $f(u) = 1 - e^u$ satisfies the condition of Corollary C.

6. Example of chaos for any small Δt. We will show an interesting example of which the Euler's difference equation is always chaotic. The original differential equation is called "extinction" (that to say, from any initial point, the solution goes into a stable equilibrium point for some finite time). It is a very strong stability. But once we change it into the difference equation, the equilibrium point becomes super unstable (derivative is infinity).

Corollary B'. Assume $f(u)$ is as follows:

$$f(u) = \begin{cases} (-u)^\alpha & (u < 0) \quad (0 < \alpha < 1) \\ -Lu^\alpha & (u \geq 0) \quad (L > 0). \end{cases}$$

Define $A = \alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}$ and let N be the unique positive solution of $N = N^\alpha + \alpha^{\frac{1}{1-\alpha}}$.
If L holds $A + N \leq LA^\alpha$, then for any $\Delta t > 0$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Corollary B'. Much the same as the proof of Corollary B. Take b such that $F_{\Delta t}(x)$ is maximum at b. $b = -(\alpha\Delta t)^{\frac{1}{1-\alpha}}$. An a exists uniquely and $a = -N\Delta t^{\frac{1}{1-\alpha}}$. For c, $c = F_{\Delta t}(b) = (\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}})\Delta t^{\frac{1}{1-\alpha}}$. Set $A = \alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}$, then $0 < A < 1$, hence $c > 0$. For d, $d = F_{\Delta t}(c) = (A - LA^\alpha)\Delta t^{\frac{1}{1-\alpha}}$. If $d \leq a$ that is $A - LA^\alpha \leq -N$, $F_{\Delta t}$ is chaotic for any $\Delta t > 0$. An invariant finite interval is $[(A - LA^\alpha)\Delta t^{\frac{1}{1-\alpha}}, A\Delta t^{\frac{1}{1-\alpha}}]$.

Q. E. D.

Example 4. When $\alpha = \dfrac{1}{2}$, then $A = \dfrac{1}{4}$, $N = \dfrac{3 + 2\sqrt{2}}{4}$ and $L \geq 2 + \sqrt{2}$.

Remark 2. It is also available when let all conditions above change symmetrically at origin.

References

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