Minor Summation Formula of Pfaffians and Schur Function Identities

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1. Introduction. In the paper [1], we exploited a minor summation formula of Pfaffians. The prototype of this formula is found in [6]. The merit of our formula is that, by taking various antisymmetric matrices, we obtain considerably various formulas on the summations of minors of a given rectangular matrix. Our motivation was in the use of the enumerative combinatorics and combinatorial representation theory. (See [9].) We are expecting the utility of this formula on various objects in this area. Particularly we think that the applications on the Schur function identities are important and we studied them intensively in [2]. There we obtained new proof of the formulas which are usually called Littlewood's formulas. Typical examples of Littlewood's formulas are the followings.

$$(1.1)\qquad \sum_{\lambda=(\alpha|\alpha+1)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i<j\le m} (1-x_ix_j),$$

$$(1.2)\qquad \sum_{\lambda=(\alpha|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) = \prod_{i=1}^{m} (1-x_i) \prod_{1\le i<j\le m} (1-x_ix_j),$$

$$(1.3)\qquad \sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i<j\le m} (1-x_ix_j).$$

(See [4].) For the notation see Section 2. In this paper we state some new results which are obtained after [2]. The method we use owes to [2], but we develop the method and exploit certain new identities which involve both the Schur functions and Čebyšev's polynomials. The main results of this paper are Theorems 3.1, 3.2 and 3.3. In the process of deriving these identities, the argument on the relation between (Sato's) Maya diagram and Murnaghan-Nakayama's formula on Young diagram has a crucial role.

2. Basic notation and a summation formula. In the paper [1] we exploited a minor summation formula of Pfaffians. Now we briefly review this formula.

Let r, m, n be positive integers such that $r \le m, n$. Let T be an arbitrary m by n matrix. For two sequences $\boldsymbol{i} = (i_1, \dots, i_r)$ and $\boldsymbol{k} = (k_1, \dots, k_r)$, let $T^{\boldsymbol{i}}_{\boldsymbol{k}} = T^{i_1\dots i_r}_{k_1\dots k_r}$ denote the submatrix of T obtained by picking up the rows and columns indexed by $\boldsymbol{i}$ and $\boldsymbol{k}$, respectively.

Assume $m \le n$ and let B be an arbitrary n by n antisymmetric matrix, that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. As long as B is a square antisymmetric matrix, we write $B_{\boldsymbol{i}} = B_{i_1\dots i_r}$ for $B^{\boldsymbol{i}}_{\boldsymbol{i}} = B^{i_1\dots i_r}_{i_1\dots i_r}$ in abbreviation. One of the main result in [1] is the following theorem. (See Theorem 1 of [1].)

Theorem 2.1. *Let $m \le n$ and $T = (t_{ik})$ be an arbitrary m by n matrix. Let m be even and $B = (b_{ik})$ be any n by n antisymmetric matrix with entries b_{ik}. Then*

$$(2.1)\qquad \sum_{1\le k_1<\dots<k_m\le n} \mathrm{pf}(B_{k_1\dots k_m})\det(T^{1\dots m}_{k_1\dots k_m}) = \mathrm{pf}(Q),$$

where Q is the m by m antisymmetric matrix defined by $Q = TB^tT$, i.e.

$$(2.2)\qquad Q_{ij} = \sum_{1\le k<l\le n} b_{kl}\det(T^{ij}_{kl}), \quad (1\le i,j\le m).$$

We regard the Pfaffian $\mathrm{pf}(B_{\boldsymbol{k}})$ as certain "weights" of the subdeterminants $\det(T^{1\dots m}_{k_1\dots k_m})$. By changing this antisymmetric matrix we obtain a considerably wide variation of the minor summation formula.

Now we review some basic notation. The reader can find these notation in [5]. A weakly decreasing sequence of nonnegative integers $\lambda := (\lambda_1, \cdots, \lambda_m)$ with $\lambda_1 \ge \cdots \ge \lambda_m \ge 0$ is called a *partition* of $|\lambda| = \lambda_1 + \cdots + \lambda_m$. The partition

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$\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ defined by $\lambda'_i = \#\{j : \lambda_j \geq i\}$ is called the conjugate partition of λ. Let $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda'_i}{2}$. For each cell $x = (i, j)$ in λ, the *hook-length* of λ at x is defined to be $h(x) = \lambda_i - j + \lambda'_j - i + 1$. Suppose that the main diagonal of λ consists of $r = p(\lambda)$ nodes. Let $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$ for $1 \leq i \leq r$. We sometimes denote the partition λ by $\lambda = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r) = (\alpha \mid \beta)$, which is called the Frobenius notation. If a is a nonnegative integer which doesn't coincide with any of α_i's, then let $q(\alpha, a)$ denote the number of α_i's which are bigger than a. For example, $\lambda = (5441)$ is the partition of 14 and $p(\lambda) = 3$. This partition is denoted by $\lambda = (421 \mid 310)$ in the Frobenius notation. If $\alpha = (310)$ then $q(\alpha, 2) = 1$ and $(\alpha + 1 \mid \alpha) = (421 \mid 310)$.

Let $\lambda = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r)$ be a partition expressed in the Frobenius notation. Let a and b be nonnegative integers such that $a \neq \alpha_1, \ldots, a_r$ and $b \neq \beta_1, \ldots, \beta_r$. There are some k and l such that $\alpha_k > a > \alpha_{k+1}$ and $\beta_l > b > \beta_{l+1}$. The partition $\lambda \cup (a \mid b)$ is defined by

(2.3) $$\lambda \cup (a \mid b) = (\alpha_1, \ldots, \alpha_k, a, \alpha_{k+1}, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_l, b, \beta_{l+1}, \ldots, \beta_r).$$

For example, $(421 \mid 310) \cup (0 \mid 2) = (4210 \mid 3210)$.

The Schur functions are well-known symmetric functions, which are known as the values of characters of the irreducible polynomial representations of the general linear group on a torus. But, here, we briefly review the definition of the Schur functions. Put

(2.4) $$T = \begin{pmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_m^{n-1} & \cdots & x_m & 1 \end{pmatrix},$$

for some fixed n. For a partition $\lambda := (\lambda_1, \cdots, \lambda_m)$, let $l = (l_1, \ldots, l_m) = \lambda + \delta$, where $\delta = (m-1, m-2, \ldots, 0)$. So we have $l_1 > l_2 > \cdots > l_m \geq 0$. Put $j_k = n - l_k$ for $1 \leq k \leq m$. Then we set $\alpha_l(x_1, \ldots, x_m) = a_{\lambda+\delta}(x_1, \ldots, x_m)$ to be

(2.5) $$a_{\lambda+\delta} = \det(T^{1\ldots m}_{j_1\ldots j_m}).$$

When $\lambda = 0$, a_δ is the famous Vandermonde determinant and equal to the product $\Pi_{1\leq i<j\leq m}(x_i - x_j)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, the Schur function $s_\lambda = s_\lambda(x_1, \ldots, x_m)$ corresponding to λ is defined to be

(2.6) $$s_\lambda = a_{\lambda+\delta}/a_\delta.$$

(See Chap. 1, Sec. 3 of [5].)

The polynomials defined by $T_n(x) = \cos(n \arccos x)$ are called Čebyšev's polynomials of the first kind, and, on the other hand, the polynomials $U_n(x) = \sin(n \arccos x)/\sqrt{1-x^2}$ are called Čebyšev's polynomials of the second kind. Both are known to satisfy the same recurrence formula:

$$P_{n+1}(x) - 2xP_n(x) + P_{n-1}(x) = 0.$$

The first few polynomials are easily calculated from the following recursion formula.

$$T_0(x) = 1, \quad U_0(x) = 0,$$
$$\begin{cases} T_{n+1}(x) = xT_n(x) + (x^2-1)U_n(x) \\ U_{n+1}(x) = T_n(x) + xU_n(x). \end{cases}$$

3. Littlewood type formulas. The following lemma is the key lemma to evaluate the pfaffian we treat.

Lemma 3.1. *Let m be a positive integer and put*

(3.1) $$Q_m(x, y) = \frac{(x^m - y^m)^2}{x-y}\frac{(1-t^m x^m y^m)^2}{1-txy}.$$

Then

(3.2) $$\mathrm{pf}[Q_m(x_i, x_j)]_{1\leq i,j\leq 2m} = \prod_{1\leq i<j\leq 2m}(x_i - x_j)(1 - tx_ix_j).$$

We fix $T = (x_i^{4m+d-2-j})_{1\leq i\leq 2m, 0\leq j\leq 4m+d-2}$ in this section. We assume $d = 2$ for a moment. Let $B = (\beta_{kl})$ be an antisymmetric matrix defined through the equation below.

(3.3) $$\sum_{0\leq k<l\leq 4m}\beta_{kl}\begin{vmatrix} x^k & x^l \\ y^k & y^l \end{vmatrix} = -(1+2ax+x^2)(1+2ay+y^2)\frac{(x^m-y^m)^2}{x-y},$$

If we apply Theorem 2.1 to Q given by the right hand side of this equation, then we obtain the following formula from Lemma 3.1 with $t = 0$.

Proposition 3.1. *Let m be a positive integer.*

(3.4) $$\sum_{k=0}^{m} U_{k+1}(a)\sum_{r=0}^{m-k} s_{(2^r1^k)}(x_1, \ldots, x_m) = \prod_{i=1}^{m}(1 + 2ax_i + x_i^2).$$

If we put $x_i = q^{2i}$ in this formula and put $m \to \infty$, then we obtain a (combinatorial) proof of the q-expansion formula of Jacobi theta functions ϑ_1 and ϑ_2, for example,

(3.5) $$\vartheta_1(u, \tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi u = 2Q_0 q^{\frac{1}{4}}\sin\pi u\prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2\pi u + q^{4n}),$$

where $q = e^{i\pi\tau}$ ($\mathrm{Im}\tau > 0$) and $Q_0 = \Pi_{n=1}^{\infty}(1 - q^{2n})$.

Let m be a positive integer and let $B=(\beta_{kl})_{0\le k,l\le m-1}$ be an antisymmetric matrix of size m in the ordinary means. Set $\boldsymbol{b}_i$ to be the i-th row vector of B for $0\le i\le m-1$. The matrix B is said to be *(row-)symmetrically proportional* if the $(m-1-k)$-th row is proportional to the k-th. That is to say, there is some c_k such that $\boldsymbol{b}_{m-1-k}=c_k\boldsymbol{b}_k$ or $\boldsymbol{b}_k=c_k\boldsymbol{b}_{m-1-k}$ for each $0\le k\le\left[\frac{m}{2}\right]-1$. Further B is called *row-symmetric* if the $\boldsymbol{b}_{m-1-k}=\boldsymbol{b}_k$ for $0\le i\le\left[\frac{m}{2}\right]-1$, and B is called *row-antisymmetric* if the $\boldsymbol{b}_{m-1-k}=-\boldsymbol{b}_k$ for $0\le k\le\left[\frac{m+1}{2}\right]-1$. This notion has importance since it makes us possible to find all the subpfaffians $\mathrm{pf}(B_{j_1\cdots j_m})$ of B. From now on we assume that B is always supposed to be antisymmetric matrix in the ordinary means.

Let $P(x)=a_0+a_1x+\cdots+a_dx^d$ be a polynomial of degree d. $P(x)$ is said to be *symmetric* if $a_i=a_{d-i}$ for $0\le i\le\left[\frac{d}{2}\right]$, and $P(x)$ is said to be *antisymmetric* if $a_i=-a_{d-i}$ for $0\le i\le\left[\frac{d+1}{2}\right]$. Then we have

Lemma 3.2. *Let $P(x)$ be a polynomial of degree d. Let $B=(\beta_{kl})_{0\le k,l\le 4m+d-2}$ be the antisymmetric matrix of size $(4m+d-1)$ which satisfy*

$$\sum_{0\le k<l\le 4m+d-2}\beta_{kl}\begin{vmatrix}x^k & x^l\\ y^k & y^l\end{vmatrix}=-P(x)P(y)Q_m(x,y). \tag{3.6}$$

The matrix B becomes (row-)symmetrically proportional for all m if and only if $P(x)$ is symmetric or antisymmetric. Further, if the polynomial $P(x)$ is symmetric then B becomes row-symmetric, on the other hand, if $P(x)$ is antisymmetric then B becomes row-antisymmetric.

From now we apply Theorem 2.1 to this T and B given by (3.6). Basically it is possible to find some sort of formula for each antisymmetric matrix B of the form (3.6) if it is row-symmetric or row-antisymmetric. Here we investigate each formula for small d. When $d=0$, we obtain the formula (1.1) from this argument. If $d=1$ and $P(x)$ is antisymmetric, we obtain the formula (1.2). It is easy to see that the case of $d=1$ and $P(x)$ being symmetric reduces to this case. If $d=2$ and $P(x)$ is antisymmetric, then we obtain the formula (1.3). These are the known Littlewood type formulas. If we assume $d=2$ and $P(x)$ is symmetric, then we obtain the following theorem.

Theorem 3.1. *Let m be a positive integer. Then*

$$\begin{aligned}&\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{\frac{|\lambda|}{2}+p(\lambda)}s_\lambda(x_1,\dots,x_m)\\ &+2\sum_{k=1}^{m}T_k(a)\sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha\not\ni k-1}}(-1)^{\frac{|\lambda|}{2}+q(\lambda,k-1)}\\ &\qquad\times s_{\lambda\uplus(0|k-1)}(x_1,\dots,x_m)\\ &=\prod_{i=1}^{m}(1+2ax_i+x_i^2)\prod_{1\le i<j\le m}(1-x_ix_j).\end{aligned}\tag{3.7}$$

If we put $x_i=q^{2i}$ in this formula and we use the q-expansion formula of Jacobi theta function ϑ_3, we obtain the following corollary.

Corollary 3.1.

$$\begin{aligned}&\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{\frac{|\lambda|}{2}+p(\lambda)}q^{\frac{|\lambda|}{2}+n(\lambda)}\prod_{x\in\lambda}\frac{1}{1-q^{h(x)}}\\ &=\frac{\prod_{r=2}^{\infty}(1-q^r)^{\left[\frac{r}{2}\right]}}{\prod_{r=1}^{\infty}(1-q^r)}.\end{aligned}\tag{3.8}$$

Let m be a nonnegative integer. Then

$$\begin{aligned}&\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{\frac{|\lambda|}{2}+q(\alpha,m)}q^{\frac{|\lambda|}{2}+n(\lambda\cup(0|m))}\\ &\times\prod_{x\in\lambda\uplus(0|m)}\frac{1}{1-q^{h(x)}}=q^{\frac{m(m+1)}{2}}\frac{\prod_{r=2}^{\infty}(1-q^r)^{\left[\frac{r}{2}\right]}}{\prod_{r=1}^{\infty}(1-q^r)}.\end{aligned}\tag{3.9}$$

If $d=3$ and $P(x)$ is antisymmetric, we obtain the following theorem. The case of $d=3$ and $P(x)$ being symmetric essentialy reduces to this case.

Theorem 3.2. *Let m be a positive integer. Then*

$$\begin{aligned}&\sum_{\lambda=(\alpha+2|\alpha)}(-1)^{\frac{|\lambda|-p(\lambda)}{2}}s_\lambda(x_1,\dots,x_m)\\ &+\sum_{k=1}^{m}\{T_k(a)+(a-1)U_k(a)\}\\ &\times\sum_{\substack{\lambda=(\alpha+2|\alpha)\\ \alpha\not\ni k-1}}(-1)^{\frac{|\lambda|+p(\lambda)}{2}+q(\lambda,k-1)}\\ &\times\{s_{\lambda\uplus(0|k-1)}(x_1,\dots,x_m)-s_{\lambda\uplus(1|k-1)}(x_1,\dots,x_m)\}\\ &=\prod_{i=1}^{m}(1+2ax_i+x_i^2)\prod_{i=1}^{m}(1-x_i)\prod_{1\le i<j\le m}(1-x_ix_j).\end{aligned}\tag{3.10}$$

If $d=4$ and $P(x)$ is antisymmetric, we obtain the following theorem.

Theorem 3.3. *Let m be a positive integer. Then*

$$\begin{aligned}&\sum_{\lambda=(\alpha+3|\alpha)}(-1)^{\frac{|\lambda|}{2}+p(\lambda)}s_\lambda(x_1,\dots,x_m)\\ &+\sum_{k=1}^{m}U_{k+1}(a)\sum_{\substack{\lambda=(\alpha+3|\alpha)\\ \alpha\not\ni k-1}}(-1)^{\frac{|\lambda|}{2}+q(\lambda,k-1)}\end{aligned}\tag{3.11}$$

$$\times \{s_{\lambda\cup(0|k-1)}(x_1,\ldots,x_m) - s_{\lambda\cup(2|k-1)}(x_1,\ldots,x_m)\}$$
$$= \prod_{i=1}^{m} (1 + 2ax_i + x_i^2) \prod_{1\leq i\leq j\leq m} (1 - x_i x_j).$$

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References

[1] M. Ishikawa and M. Wakayama: Minor summation of Pfaffians (to appear in Linear and Multilinear Alg.).

[2] M. Ishikawa, S. Okada and M. Wakayama: Applications of minor summation farmulas I, Littlewood's formulas (preprint).

[3] C. Krattenthaler: On bideterminantal formulas for characters of classical groups (preprint).

[4] K. Koike and I. Terada: Littlewood's formulas and their application to representations of classical groups. Advanced Studies in Pure Math., **11**, 147–160 (1987).

[5] I. G. Macdonald: Symmetric Functions and Hall Polynomials. Oxford University Press (1979).

[6] S. Okada: On the generating functions for certain classes of plane partitions. J. Combin. Theorey, ser. A, **51**, 1–23 (1989).

[7] R. A. Proctor: Young Tableaux, Gelfand Patterns, and Branching Rules for Classical Groups(preprint).

[8] R. P. Stanley: Theory and application of plane partitions. I. II. Studies appl. Math., **50**, 167–188, 259–279 (1971).

[9] R. P. Stanley: Enumerative Combinatorics. vol. I, Wadsworth, Monterey CA (1986).

[10] J. Stembridge: Nonintersecting paths and pfaffians. Adv. in Math., **83**, 96–131 (1990).