On Homology and Cohomology of Lie Superalgebras with Coefficients in Their Finite-Dimensional Representations

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In this paper we discuss explicit calculations of homology and cohomology of a Lie superalgebra. Complete results fore $\mathfrak{gl}(1,1)$ and $\mathfrak{sl}(2,1)$ are given in case the dimensions of representations are finite. Our result implies that for any $n \in \boldsymbol{Z}_{\geq 0}$, there exists a finite-dimensional irreducible $\mathfrak{g}$-module V such that $\mathbf{H}^n(\mathfrak{g}, V) \neq \{0\}$, contrary to the case of finite-dimensional Lie algebras. This means that the Poincaré duality, which is proved by S.Chemla [1] under a certain restrictive condition, does not hold in general in our case. For definitions and notations we mainly follow Kac [6].

1. Generalities. Homology groups $\mathbf{H}_n(\mathfrak{g}, V)$ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with coefficients in its representation space V are defined similarly as for a Lie algebra (cf. [7, p. 283]) and they can be obtained as $\operatorname{Ker} \partial_{n-1} / \operatorname{Im} \partial_n$ in the following complex (B, ∂) :

$$0 \leftarrow B_0 \overset{\partial_0}{\leftarrow} B_1 \overset{\partial_1}{\leftarrow} B_2 \overset{\partial_2}{\leftarrow} B_3 \overset{\partial_3}{\leftarrow} \cdots, \quad B_n = \wedge^n \mathfrak{g} \otimes V,$$

$$\begin{aligned}\partial_{n-1}&(X_1 \wedge \cdots \wedge X_n \otimes v)\\ &= \sum_{i=1}^{n} (-1)^{i+\eta'_i} X_1 \wedge \cdots \hat{i} \cdots \wedge X_n \otimes X_i v\\ &\quad + \sum_{k<l} (-1)^{k+l+\eta_k+\eta_l+\xi_k\xi_l} [X_k, X_l]\\ &\qquad\qquad \wedge X_1 \wedge \cdots \hat{k} \cdots \hat{l} \cdots \wedge X_n \otimes v,\end{aligned}$$

where $X_i \in \mathfrak{g}$ homogeneous, $v \in V$, $\xi_i = |X_i| := \deg X_i$, $\eta_i = \xi_i(\xi_1 + \cdots + \xi_{i-1})$, $\eta'_i = \xi_i(\xi_{i+1} + \cdots + \xi_n)$, and the symbol $\hat{i}$ indicates a term X_i to be omitted (cf. [8]). The Grassmann algebra $\wedge \mathfrak{g}$ here is defined as the quotient of the tensor algebra of $\mathfrak{g}$ by a two-sided ideal generated by $\{X \otimes Y + (-1)^{|X||Y|} Y \otimes X \mid X, Y \in \mathfrak{g}$; homogeneous$\}$ and it is a $\mathfrak{g}$-module through a natural action:

$$\begin{aligned}&X \cdot (X_1 \wedge \cdots \wedge X_n)\\ &= \sum (-1)^{|X|(\xi_1+\cdots+\xi_{i-1})} X_1 \wedge \cdots \wedge [X, X_i] \wedge \cdots \wedge X_n.\end{aligned}$$

Then B_n's are $\mathfrak{g}$-modules with $\rho_n(X)(\theta \otimes v) = X\theta \otimes v + (-1)^{|X||\theta|} \theta \otimes Xv$ $(X \in \mathfrak{g}, \theta = X_1 \wedge \cdots \wedge X_n \in \wedge^n \mathfrak{g}, |\theta| = \xi_1 + \cdots + \xi_n, v \in V)$. This action commutes with the derivation ∂, that is, $X \circ \partial_n = \partial_{n-1} \circ X$.

We appeal to the following lemmas to calculate the homology and the cohomology.

Lemma 1. *Let $\mathfrak{q}$ be a subalgebra of $\mathfrak{g}$ such that its natural representation $\rho_n|_{\mathfrak{q}}$ on the n-th chain B_n are all semisimple. Then, the homology $\mathbf{H}_n(\mathfrak{g}, V)$ can be obtained from a subcomplex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$, where the n-th chain $B_n^{\mathfrak{q}}$ for $B^{\mathfrak{q}}$ is the subspace of $\mathfrak{q}$-invariants in B_n.*

The space $V^* := \operatorname{Hom}_{\boldsymbol{C}}(V, \boldsymbol{C})$ has a natural $\mathfrak{g}$-module structure.

Lemma 2 (Duality). *Let $\mathfrak{g}$ be a Lie superalgebra and V a $\mathfrak{g}$-module. Assume that $\mathfrak{g}$ and V are both finite-dimensional, then there are $\mathfrak{g}$-module isomorphisms between homology groups and cohomology groups as*

$$\mathbf{H}^n(\mathfrak{g}, V^*) \cong \mathbf{H}_n(\mathfrak{g}, V)^*.$$

2. Case of $\mathfrak{gl}(1,1)$. Fix a basis of the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1, 1)$ as follows:

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The elements H and C generate a Cartan subalgebra, which is equal to the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ in this simplest case. Put $\mathfrak{g}_1 = \boldsymbol{C}X$ and $\mathfrak{g}_{-1} = \boldsymbol{C}Y$. Then the odd part is $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_1 + \mathfrak{g}_{-1}$, and this gives a $\boldsymbol{Z}$-grading of $\mathfrak{g}$ together with $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$. Let $L(\Lambda) := \boldsymbol{C}v_0$ be a one-dimensional representation of $\mathfrak{g}_{\bar{0}}$ given by $Hv_0 = \lambda v_0$, $Cv_0 = cv_0$ $(\lambda, c \in \boldsymbol{C})$ and Λ denote a pair (λ, c). For a subalgebra $\mathfrak{p} := \mathfrak{g}_{\bar{0}} + \mathfrak{g}_1$, we extend $L(\Lambda)$ as a $\mathfrak{p}$-module through a trivial action of X. Then the induced module $\bar{V}(\Lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} L(\Lambda)$ defines a representation of $\mathfrak{g}$. $\bar{V}(\Lambda)$ is irreducible if and only if $c \neq 0$.

We calculate the homology $\mathbf{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$, which is isomorphic to $\mathbf{H}_n(\mathfrak{p}, L(\Lambda))$ by Shapiro's lemma on induced modules (cf. [7]). Put $X^{(k)} = X \wedge X \wedge \cdots \wedge X \in \wedge^k \mathfrak{g}$ and

$$\alpha_n = X^{(n)} \otimes v_0,\ \beta_n = H \wedge X^{(n-1)} \otimes v_0,$$
$$\gamma_n = C \wedge X^{(n-1)} \otimes v_0,\ \delta_n = C \wedge H \wedge X^{(n-2)} \otimes v_0.$$

Then, they generate the space B_n of n-th chains. Now we take $\mathfrak{g}_{\bar{0}}$ as a subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ in Lemma 1. It is necessary that $c = 0$ and $\lambda \in -\boldsymbol{Z}_{\geq 0}$ for a subcomplex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$ to be non-trivial. In that case, the complex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$ in Lemma 1 for $V = \bar{V}(\Lambda)$ can be written as

$$0 \leftarrow \boldsymbol{C}\alpha_{-\lambda} \overset{\partial}{\leftarrow} \langle \beta_{1-\lambda}, \gamma_{1-\lambda} \rangle_{\boldsymbol{C}} \overset{\partial}{\leftarrow} \boldsymbol{C}\delta_{2-\lambda} \leftarrow 0,$$

and the derivation ∂ is equal to zero. For calculation of cohomology groups we use the duality in Lemma 2 and $\bar{V}(\lambda, 0)^* \cong \bar{V}(-\lambda, 0)$. Thus we have the homology $\mathbf{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$ and cohomology $\mathbf{H}^n(\mathfrak{g}, \bar{V}(\Lambda))$ as in the following theorem.

Theorem 3. *In case $c = 0$ and $\lambda \in -\boldsymbol{Z}_{\geq 0}$, $\dim \mathbf{H}_n(\mathfrak{g}, \bar{V}(\Lambda)) = 1$ $(n = -\lambda, -\lambda + 2)$, and $= 2$ $(n = -\lambda + 1)$. In all other cases, $\mathbf{H}_n(\mathfrak{g}, \bar{V}(\Lambda)) = \{0\}$.*

In case $c = 0$ and $\lambda \in \boldsymbol{Z}_{\geq 0}$, $\dim \mathbf{H}^n(\mathfrak{g}, \bar{V}(\Lambda)) = 1$ $(n = \lambda, \lambda + 2)$, and $= 2$ $(n = \lambda + 1)$. In all other cases, $\mathbf{H}^n(\mathfrak{g}, \bar{V}(\Lambda)) = \{0\}$.

In case $c = 0$, the module $\bar{V}(\Lambda)$ is reducible and has a unique maximal proper submodule, say $I(\Lambda)$, and the quotient is a unique (up to isomorphisms) irreducible representation $V(\Lambda)$ of $\mathfrak{gl}(1,1)$ with the highest weight $\Lambda : V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda)$. By calculations, we get the following result (cf. [9]).

Theorem 4. *Let $c = 0$. If $\lambda \in -\boldsymbol{Z}_{\geq 0}$, then, $\dim \mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = \dim \mathbf{H}^n(\mathfrak{g}, V(\Lambda)) = 1$ $(n = -\lambda, -\lambda + 1)$ and $\mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = \mathbf{H}^n(\mathfrak{g}, V(\Lambda)) = \{0\}$ otherwise.*

If $\lambda \in \boldsymbol{Z}_{>0}$, then, $\dim \mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = \dim \mathbf{H}^n(\mathfrak{g}, V(\Lambda)) = 1$ $(n = \lambda, \lambda + 1)$ and $\mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = \mathbf{H}^n(\mathfrak{g}, V(\Lambda)) = \{0\}$ otherwise.

3. Case of $\mathfrak{sl}(2,1)$. Let

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and $Z_+ = E_{12}$, $Z_- = E_{21}$, $X_i = E_{i3}$, $Y_i = E_{3i}$ $(i = 1,2)$, where E_{ij} denotes the elementary matrix with 1 in (i, j)-component and 0 elsewhere. The elements H, Z_+ and Z_- generate a Lie algebra which may be written as $\mathfrak{sl}(2, \boldsymbol{C})$. We take an irreducible representation $V_0 = L(\Lambda)$ of $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2, \boldsymbol{C}) \oplus \boldsymbol{C} \cdot C$ with $\Lambda = (\lambda, c)$, which is a $(\lambda + 1)$-dimensional irreducible $\mathfrak{sl}(2, \boldsymbol{C})$-module $(\lambda \in \boldsymbol{Z}_{\geq 0})$ and on which C acts as a scalar multiple by $c \in \boldsymbol{C}$. We get an induced representation $\bar{V}(\Lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} V_0$, where $\mathfrak{p} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_1$ and $\mathfrak{g}_1 = \langle X_1, X_2 \rangle_{\boldsymbol{C}}$. Define $V(\Lambda)$ as an irreducible quotient of $\bar{V}(\Lambda)$ by a maximal submodule $I(\Lambda)$ of $\bar{V}(\Lambda)$. Every finite-dimensional irreducible representation of $\mathfrak{sl}(2,1)$ is realized as $V(\Lambda)$. $\bar{V}(\Lambda)$ is irreducible if and only if $(\lambda - c)(\lambda + c + 2) \neq 0$. In case $\bar{V}(\Lambda)$ is irreducible, we can get the homology groups $\mathbf{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$ which are isomorphic to $\mathbf{H}_n(\mathfrak{p}, L(\Lambda))$ by Shapiro's lemma. The latter vanish for any n.

CASE $\lambda = c \in \boldsymbol{Z}_{\geq 0}$. When $\lambda = c = 0$, $V(\lambda, c) = \mathbf{C}$ and homology groups are obtained similarly to the follwing case.

In case $\lambda = c > 0$, we have $V(\Lambda) \cong I(\Lambda')$ with $\Lambda' := (\lambda', c') = (\lambda - 1, c - 1)$, and $I(\Lambda')$ is decomposed into two irreducible $\mathfrak{g}_{\bar{0}}$-modules (cf. [4]) as $I(\Lambda') = I_1 \oplus I_2$ with $I_1 := \langle -i(Y_1 \otimes v_{i-1}) + Y_2 \otimes v_i \mid 0 \leq i \leq \lambda' + 1 \rangle_{\boldsymbol{C}}$ and $I_2 := Y_1 Y_2 \otimes L(\Lambda)$. Accordingly we have $B_n = B_n^{\ 1} \otimes B_n^{\ 2}$, where $B_n^{\ i} = \oplus_{p+q+r=n} (\wedge^p \mathfrak{g}_{\bar{0}} \otimes \wedge^q \mathfrak{g}_1 \otimes \wedge^r \mathfrak{g}_{-1} \otimes I_i)$. We take $\mathfrak{g}_{\bar{0}}$ as $\mathfrak{q}$ in Lemma 1. On each component, C acts as a scalar multiple by $-q + r + i$. $\wedge \mathfrak{g}_{\bar{0}}$ is decomposed into four 3-dimensional irreducible $\mathfrak{sl}(2, \boldsymbol{C})$-modules and a 4-dimensional trivial $\mathfrak{sl}(2, \boldsymbol{C})$-module, while highest weights of $\wedge^q \mathfrak{g}_1$, $\wedge^r \mathfrak{g}_{-1}$ and I_i are q, r and $\lambda' + 2 - i$ respectively. Here $\mathfrak{g}_{-1} = \langle Y_1, Y_2 \rangle_{\boldsymbol{C}}$.

Lemma 5. *Let $V_n (n \in \boldsymbol{Z}_{\geq 0})$ denote an $(n + 1)$-dimensional irreducible $\mathfrak{sl}(2, \boldsymbol{C})$-module. For k, $l \in \boldsymbol{Z}_{\geq 0}$, the tensor product of two modules V_k and V_l is a direct sum of $\min(k, l) + 1$ number of $\mathfrak{sl}(2, \boldsymbol{C})$-modules as $V_k \otimes V_l = \oplus_{j=0}^{\min(k,l)} V_{k+l-2j}$.*

Using this well-known lemma, we see that $(B_n^{\ 1})^{\mathfrak{q}}$ and $(B_n^{\ 2})^{\mathfrak{q}}$ are 6- and 2-dimensional spaces respectively for sufficiently large n and that for some small n's, dimensions of $B_n^{\ \mathfrak{q}}$ are smaller than $8 = 6 + 2$. Fix explicitly a basis of $B_n^{\ \mathfrak{q}}$, and compute ∂ on them, then we can obtain the next table:

n	$\lambda' + 1$	$\lambda' + 2$	$\lambda' + 3$	$\lambda' + 4$	$\lambda' + 5$	$\lambda' + 6$	......
$\dim D_n$	1	2	4	7	8	8	......
$\dim \mathrm{Ker}\ \partial_{n-1}$	1	2	2	5	4	4	......
$\dim \mathrm{Im}\ \partial_n$	0	2	2	4	4	4	......

From this result, we have the following proposition.

Proposition 6. *Let $\Lambda' = (\lambda', c')$ with $\lambda' = c' \in \boldsymbol{Z}_{\geq 0}$. Then dimensions of homology groups of irreducible* $\mathfrak{g}$*-module* $I(\Lambda')$ *are*

$$\dim \mathbf{H}_n(\mathfrak{g}, I(\Lambda')) = 1 \ (n = \lambda' + 1, \lambda' + 4), \quad \textit{and} = 0 \ (\textit{otherwise}).$$

CASE $\lambda = -c - 2 \in \boldsymbol{Z}_{\geq 0}$. In this case, $V(\Lambda) \cong I(\Lambda')$ with $\Lambda' = (\lambda', c') = (\lambda + 1, c - 1)$, and $I(\Lambda') = I_1' \otimes I_2'$ with $I_1' := \langle (\lambda' - i) Y_1 \otimes v_i + Y_2 \otimes v_{i+1} \mid 0 \leq i \leq \lambda' + 1 \rangle_{\boldsymbol{C}}$ and $I_2' := Y_1 Y_2 \otimes L(\Lambda)$. The calculations are similar and we get the following.

Proposition 7. *Let $\Lambda' = (\lambda', c')$ with $\lambda' = -c' - 2 \in \boldsymbol{Z}_{\geq 0}$. Then dimensions of homology groups of irreducible module* $I(\Lambda')$ *are*

$$\dim \mathbf{H}_n(\mathfrak{g}, I(\Lambda')) = 1 \ (n = \lambda', \lambda' + 3), \quad \textit{and} = 0 \ (\textit{otherwise}).$$

We get our main result for $\mathfrak{sl}(2,1)$ from these propositions and the duality in Lemma 2 and $V(\lambda, c)^* \cong V(\lambda', c')$ with $\lambda' = \lambda - 1$, $c' = -c - 1$ in case $\lambda = c > 0$ (and so $\lambda' + c' + 2 = 0$).

Theorem 8. *Let $V(\Lambda)$ be a finite-dimensional irreducible representation of $\mathfrak{g} = \mathfrak{sl}(2,1)$ with highest weight $\Lambda = (\lambda, c)$, $\lambda \in \boldsymbol{Z}_{\geq 0}$, $c \in \boldsymbol{C}$. Then, in case $\lambda = c$,*

$$\dim \mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = \dim \mathbf{H}^n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & (n = \lambda, \lambda + 3) \\ 0 & (\textit{otherwise}). \end{cases}$$

In case $\lambda + c + 2 = 0$,

$$\dim \mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = \dim \mathbf{H}^n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & (n = \lambda + 1, \lambda + 4) \\ 0 & (\textit{otherwise}). \end{cases}$$

In case $(\lambda - c)(\lambda + c + 2) \neq 0$,

$$\mathbf{H}_n(\mathfrak{g}, V(\Lambda)) = 0 \ \textit{for any } n.$$

The details for $\mathfrak{g} = \mathfrak{sl}(2,1)$ will appear elsewhere [10].

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