A Mean Value Property in Adele Geometry

By Masanori MORISHITA

Department of Mathematics, Kanazawa University

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1995)

Introduction. Let X be a left homogeneous space of a connected linear algebraic group G. Suppose that G, X and the action are defined over $\boldsymbol{Q}$, the field of rational numbers, and that X has a $\boldsymbol{Q}$-rational point x. We then identify X with G/H, where H is the stabilizer of x.

After the works of Siegel [13] and Weil [14], Ono [10] investigated a mean value theorem for the adele space attached to a *uniform* and *special* homogeneous space $X = G/H$, introducing the Tamagawa number $\tau(G, X)$. Here, X is said to be *special* if G and H are connected linear $\boldsymbol{Q}$-groups without tori parts in their Levi-Chevalley decompositions.

In [8], using Kottwitz's fundamental theorem on the Tamagawa number [6], we showed that any special homogeneous space is uniform, and gave a formula expressing $\tau(G, X)$ in terms of the fundamental groups of G and H.

The purpose of this paper is to give a generalization of our results for special homogeneous spaces to those for a wider class of homogeneous spaces allowing G and H to have $\boldsymbol{Q}$-anisotropic tori in their Levi-Chevalley decompositions. Since a reductive group does not have a universal covering in general, we use Borovoi's algebraic fundamental group to describe our results. Also, we use his theory on abelian Galois cohomology which is a machinery to study Galois cohomology of connected linear algebraic groups in a functorial way ([1], [2], [3] and Appendix B to [7]).

I am grateful to Mikhail V. Borovoi for useful comments on the earlier version of this paper. I also thank Takao Watanabe for the correspondence on [8].

1. Borovoi's fundamental group and abelian Galois cohomology. In this section, we introduce Borovoi's algebraic fundamental group and abelian Galois cohomology which we need later to describe our results. For these matters, we refer to [1], [2], [3], and also Appendix B to [7].

Let k be a field of characteristic zero and $\bar{k}$ a fixed algebraic closure of k. First, we assume that G is reductive. Let G^{ss} be the derived group of G and G^{sc} be the universal k-covering of G^{ss} [9], Appendix I). Consider the composition

$$\rho : G^{sc} \to G^{ss} \subset G.$$

Take a maximal torus T in $G_{\bar{k}}$ and put $T^{sc} = \rho^{-1}(T)$. We then define

$$\pi_1(G, T) := X_*(T)/\rho_* X_*(T^{sc}),$$

where $X_*(S)$ denotes the group of one-parameter subgroups of a torus S. If T' is another maximal torus in $G_{\bar{k}}$, there is $g \in G(\bar{k})$ so that $T' = gTg^{-1} = Int(g)(T)$. Then, $Int(g)$ induces the isomorphism $g_* : \pi_1(G, T) \simeq \pi_1(G, T')$ which does not depend on the choice of g. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $\pi_1(G, T)$ in the following way. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, there is $g_\sigma \in G(\bar{k})$ so that $T^\sigma = g_\sigma^{-1} T g_\sigma$. Then, σ acts on $\pi_1(G, T)$ as the composition

$$\pi_1(G, T) \xrightarrow{\sigma_*} \pi_1(G, T^\sigma) \xrightarrow{(g\sigma)_*} \pi_1(G, T).$$

We see that the above isomorphism g_* is $\mathrm{Gal}(\bar{k}/k)$-equivariant. So, we simply write $\pi_1(G)$ for this Galois module. For a connected linear k-group G, we set $\pi_1(G) := \pi_1(G/G^u)$, where G^u is the unipotent radical of G, and call it *Borovoi's fundamental group* of G. Then, $\pi_1(\cdot)$ is an exact functor from the category of connected linear k-groups to $\mathrm{Gal}(\bar{k}/k)$-modules, finitely generated over $\boldsymbol{Z}$. One sees that an inner twisting $G \to G'$ induces the isomorphism $\pi_1(G) \simeq \pi_1(G')$, and that if $k \subset \boldsymbol{C}$, $\pi_1(G)$ is canonically isomorphic to the topological fundamental group of the complex Lie group $G(\boldsymbol{C})$ as abelian groups.

Next, we define the abelian Galois cohomology groups of a connected reductive group G by

$$H^i_{ab}(k, G) := \boldsymbol{H}^i(k, T^{sc} \to T) \ (i \geq -1),$$

where $\boldsymbol{H}^i$ means the Galois hypercohomology of the complex

$$0 \to T^{sc} \to T \to 0,$$

where T^{sc} and T sit in degree -1 and 0, respectively.

Noting that $(X_*(T^{sc}) \overset{\rho_*}{\to} X_*(T)) \to \pi_1(G)$ is a short torsion free resolution of $\pi_1(G)$ and that $S(\bar{k}) = X_*(S) \otimes \bar{k}^\times$ for a k-torus S, we can see that $H^i_{ab}(k, G)$ depends only on the Galois module $\pi_1(G)$. For a connected k-group G, we set $H^i_{ab}(k, G) := H^i_{ab}(k, G/G^u)$.

On the other hand, for a connected reductive group G, we observe that $\rho : G^{sc} \to G$ is a crossed module of algebraic groups over k and so we can also define, in terms of cocycles, the hypercohomology

$$\boldsymbol{H}^i(k, G^{sc} \to G)$$

for $i = -1,0,1$, in a functorial way. Then, using the morphism $(1 \to G) \to (G^{sc} \to G)$ and the quasi-isomorphism $(T^{sc} \to T) \to (G^{sc} \to G)$ of crossed modules, we define the abelianization maps

$$ab^i : H^i(k, G) \to H^i_{ab}(k, G)$$

for $i = 0,1$ (For ab^2, see [2]). For a connected k-group G, the abelianization maps are defined by the composition

$$H^i(k, G) \to H^i(k, G/G^u) \overset{ab^i}{\to} H^i_{ab}(k, G/G^u) = H^i_{ab}(k, G).$$

Finally, we remark that if G is semisimple, $\pi_1(G) = \mathrm{Ker}(\tilde{\rho})(-1)$ (Tate twist), $H^i_{ab}(k, G) = H^{i+1}(k, \mathrm{Ker}\rho)$ and ab^i $(i = 0,1)$ are connecting homomorphisms attached to the exact sequence, $1 \to \mathrm{Ker}\rho \to G^{sc} \to G \to 1$.

2. Global and local classes. Let X be a left homogeneous space over $\boldsymbol{Q}$ of a connected linear $\boldsymbol{Q}$-group G, and H the stabilizer of a $\boldsymbol{Q}$-rational point x of X. We define two equivalence relation on the set $X(\boldsymbol{Q})$. Let y, z be in $X(\boldsymbol{Q})$. We say that y is globally equivalent to z, written $y \sim z$, if there is $g \in G(\boldsymbol{Q})$ so that $z = gy$, and y is locally equivalent to z if there is $g_{\boldsymbol{A}} \in G(\boldsymbol{A})$ so that $z = g_{\boldsymbol{A}} y$ in $X(\boldsymbol{A})$, where $\boldsymbol{A}$ denotes the adele ring of $\boldsymbol{Q}$. Thus, the local class Θ_x containing x is $G(\boldsymbol{A})x \cap X(\boldsymbol{Q})$.

For a connected linear $\boldsymbol{Q}$-group G, we define $\mathrm{Ker}^1(\boldsymbol{Q}, G)$ to be the kernel of the localization map

$$H^1(\boldsymbol{Q}, G) \to \prod_v H^1(\boldsymbol{Q}_v, G),$$

where v runs over all places of $\boldsymbol{Q}$, and $\boldsymbol{Q}_v$ is the completion of $\boldsymbol{Q}$ at v. Similarly, we define

$$\mathrm{Ker}^1_{ab}(\boldsymbol{Q}, G) := \mathrm{Ker}(H^1_{ab}(\boldsymbol{Q}, G) \to \prod_v H^1_{ab}(\boldsymbol{Q}_v, G)).$$

The following theorem, due to Borovoi, is a natural generalization of [12], Theorem 4.3.

Theorem 2.1. ([3]) *The abelianization map $ab^1 : H^1(\boldsymbol{Q}, G) \to H^1_{ab}(\boldsymbol{Q}, G)$ induces a bijection of $\mathrm{Ker}^1(\boldsymbol{Q}, G)$ onto the finite abelian group $\mathrm{Ker}^1_{ab}(\boldsymbol{Q}, G)$, which is functorial in G.*

In particular, $\mathrm{Ker}^1(\boldsymbol{Q}, G)$ depends only on the Galois module $\pi_1(G)$.

Combining Theorem 2.1 with Lemma 2.1 of [8], we have

Theorem 2.2. *Notation being as above, we have a bijection*

$$\Theta_x/\sim \;\simeq \mathrm{Ker}\,(\mathrm{Ker}^1_{ab}(\boldsymbol{Q}, H) \to \mathrm{Ker}^1_{ab}(\boldsymbol{Q}, G)).$$

In particular, the cardinality of the set $\Theta_x/\sim$ does not depend on $x \in X(\boldsymbol{Q})$.

We write $h(G, X)$ for the cardinality of $\Theta_x/\sim$, $x \in X(\boldsymbol{Q})$.

3. Tamagawa number and mean value property. Let G, H and X be as in Section 2. Assume further that G, H are unimodular connected $\boldsymbol{Q}$-groups. Then, we have invariant gauge forms ω^G, ω^H on G, H and G-invariant gauge form ω^X on X so that they match together algebraically, $\omega^G = \omega^X\omega^H$ ([15], 2.4). For each place v of $\boldsymbol{Q}$, these gauge forms induce the local measures ω^G_v, ω^H_v and ω^X_v on $G(\boldsymbol{Q}_v)$, $H(\boldsymbol{Q}_v)$ and $X(\boldsymbol{Q}_v)$, respectively, which match together topologically ([15], 2.4). The Tamagawa measure $\omega^G_{\boldsymbol{A}}$ on $G(\boldsymbol{A})$ is defined by

$$\omega^G_{\boldsymbol{A}} = \rho_G^{-1} \prod_v L_v(1, X^*(G))\omega^G_v,$$

where $L_v(s, X^*(G))$ is the v-factor of the Artin L-function $L(s, X^*(G))$ attached to the representation of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ on the module $X^*(G)$ of rational characters of G, $\rho_G = lim_{s\to 1}(s-1)^{r_G} L(s, X^*(G))$,

r_G is the rank of the submodule $X^*(G)_{\boldsymbol{Q}}$ consisting of $\boldsymbol{Q}$-rational characters of G. ([15], Appendix II). Denoting $G(\boldsymbol{A})^1$ the subgroup of all $g \in G(\boldsymbol{A})$ such that the idele norm of $\chi(g)$ is 1 for all $\boldsymbol{Q}$-rational character χ of G, we define the Tamagawa number $\tau(G)$ to be the volume of $G(\boldsymbol{A})^1/G(\boldsymbol{Q})$ with respect to the Tamagawa measure $\omega^G_{\boldsymbol{A}}$. We define the Tamagawa measure $\omega^X_{\boldsymbol{A}}$ on $X(\boldsymbol{A})$ by

$$\omega^X_{\boldsymbol{A}} = \rho_X^{-1} \prod_v \lambda_v \omega^X_v,$$

where $\rho_X = \rho_G/\rho_H$, $\lambda_v = L_v(1, X^*(G))/L_v(1, X^*(H))$.

Then, we see that the measures $\omega^G_{\boldsymbol{A}}$, $\omega^H_{\boldsymbol{A}}$ and $\omega^X_{\boldsymbol{A}}$ match together topologically.

Here, we recall the fundamental theorem on the Tamagawa number of an algebraic group in terms of Borovoi's fundamental group.

Theorem 3.1. ([6], [5]) *The Tamagawa number* $\tau(G)$ *of a unimodular connected linear* $\boldsymbol{Q}$-*group* G *is given by*

$$\tau(G)=\frac{[(\pi_1(G)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})})_{tors}]}{[\mathrm{Ker}^1(\boldsymbol{Q},G)]},$$

where $(\pi_1(G)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})})_{tors}$ *means the torsion part of the coinvariant quotient of* $\pi_1(G)$ *under* $\mathrm{Gal}(\bar{k}/k)$, *and* $[*]$ *means the cardinality of a set* $*$.

In particular, $\tau(G)$ *does not change under an inner twisting.*

Now, throughout the following, we assume further that G and H have no non-trivial $\boldsymbol{Q}$-rational characters and X is a quasi-affine $\boldsymbol{Q}$-variety. (Note that X is quasi-projective if no condition is imposed on G, H, and X becomes affine if H is reductive). Hence, $G(\boldsymbol{A})=G(\boldsymbol{A})^1$, $H(\boldsymbol{A})=H(\boldsymbol{A})^1$, and $X(\boldsymbol{Q})$ is discrete in $X(\boldsymbol{A})$.

Since $G(\boldsymbol{A})X(\boldsymbol{Q})$ is open and closed in $X(\boldsymbol{A})$, the Tamagawa measure $\omega_{\boldsymbol{A}}^X$ induces a measure, written also $\omega_{\boldsymbol{A}}^X$, on this subset. Let $L(G(\boldsymbol{A})X(\boldsymbol{Q}))$ be the set of all compactly-supported continuous functions on $G(\boldsymbol{A})X(\boldsymbol{Q})$.

Firstly, we have the following theorem for the uniformity of (G, X).

Theorem 3.2. *Notation being as above, we have an equality*:

$$h(G,X)\tau(H)\int_{G(\boldsymbol{A})X(\boldsymbol{Q})}f(x_{\boldsymbol{A}})\omega_{\boldsymbol{A}}^X=\int_{G(\boldsymbol{A})/G(\boldsymbol{Q})}\Big(\sum_{y\in X(\boldsymbol{Q})}f(g_{\boldsymbol{A}}y)\Big)\omega_{\boldsymbol{A}}^G$$

for all $f\in L(G(\boldsymbol{A})X(\boldsymbol{Q}))$.

Proof. We have only to repeat Ono's argument in Lemma 8.3 of [10], using Theorems 2.2 and 3.1, since stabilizers of $x\in X(\boldsymbol{Q})$ are inner forms each other and so their fundamental groups are $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$-isomorphic. □

Definition 3.3. We call $\tau(G,X):=\tau(G)/h(G,X)\tau(H)$ the *Tamagawa number* of a homogeneous space (G, X) and say that (G, X) has the *mean value property* if $\tau(G,X)=1$, namely, the following equality holds.

$$\int_{G(\boldsymbol{A})X(\boldsymbol{Q})}f(x_{\boldsymbol{A}})\omega_{\boldsymbol{A}}^X=\tau(G)^{-1}\times\int_{G(\boldsymbol{A})/G(\boldsymbol{Q})}\Big(\sum_{y\in X(\boldsymbol{Q})}f(g_{\boldsymbol{A}}y)\Big)\omega_{\boldsymbol{A}}^G$$

for all $f\in L(G(\boldsymbol{A})X(\boldsymbol{Q}))$.

Theorem 3.4. *If* $\mathrm{Ker}^1(\boldsymbol{Q},G)=1$, *then we have*

$$\tau(G,X)=\frac{[\pi_1(G)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}]}{[\pi_1(H)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}]}.$$

Proof. By Theorems 2.1, 2.2 and 3.1, we have only to show that $\pi_1(H)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}$ and $\pi_1(G)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}$ are finite. The finiteness of $\pi_1(H)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}$ is reduced to that of $\pi_1(H^{tor})_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}=X_*(H^{tor})_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}$, where H^{tor} is the biggest quotient torus of H. The latter follows from the assumption that H has no non-trivial $\boldsymbol{Q}$-rational characters. The finiteness of $\pi_1(G)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}$ follows in the same way. □

The following theorem is a generalization and refinement of Ono's mean value theorem ([10], Theorem 9.1).

Theorem 3.5. *If the first two homotopy groups of the complex manifold* $X(\boldsymbol{C})$ *vanish,* (G, X) *has the mean value property.*

Proof. By the assumption and the homotopy exact sequence attached to the fibration

$$1\to H(\boldsymbol{C})\to G(\boldsymbol{C})\to X(\boldsymbol{C})\to 1,$$

we have a $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$-isomorphism

$$\pi_1(H)\simeq\pi_1(G).$$

Hence, our assertion follows from Theorems 2.1, 2.2 and 3.1. □

4. Examples. Here, we give two examples where the homogeneous spaces are not special in the sense of Ono [10].

4.1. (Hopf homogeneous space). Let K be a quadratic field over $\boldsymbol{Q}$. Let $R_{K/\boldsymbol{Q}}(\mathbf{G_m})$ be the Weil restriction of the multiplicative group $\mathbf{G_m}$ from K to $\boldsymbol{Q}$ ([15], 1.3). Let $N:R_{K/\boldsymbol{Q}}(\mathbf{G_m})\to\mathbf{G_m}$ be the norm map attached to $K/\boldsymbol{Q}$. For $x\in R_{K/\boldsymbol{Q}}(\mathbf{G_m})$, define $\bar{x}\in R_{K/\boldsymbol{Q}}(\mathbf{G_m})$ by $x\bar{x}=N(x)$.

Let $G=\left\{\begin{pmatrix} x & y\\ -\bar{y} & \bar{x}\end{pmatrix}\mid N(x)+N(y)=1\right\}$,

the special unitary group ("3-sphere") attached to $K/\boldsymbol{Q}$. Denoting by S^2 the "2-sphere" $\{(z,w)\in\mathbf{G_m}\times R_{K/\boldsymbol{Q}}(\mathbf{G_m})\mid z^2+N(w)=1\}$, we have a *Hopf map* (cf. [11], Chap. 5)

$$G\to S^2,\ \begin{pmatrix} x & y\\ -\bar{y} & \bar{x}\end{pmatrix}\mapsto(N(x)-N(y),\,2xy)$$

which induces a bijection as we can show easily,

$$G/H\simeq S^2,$$

where $H=\left\{\begin{pmatrix} t & 0\\ 0 & \bar{t}\end{pmatrix}\mid N(t)=1\right\}$, which is a $\boldsymbol{Q}$-anisotropic torus.

In view of this bijection, we would like to call $X=G/H$ the *Hopf homogeneous space* attached to $K/\boldsymbol{Q}$. For its Tamagawa number, by Theorem 3.4, we have

$$\tau(G,X)=[\pi_1(H)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})}]^{-1}$$

$= [H^1(\boldsymbol{Q}, X^*(H))]^{-1} = 1/2.$

4.2. ([4], 6.6) Let $f = t^n + a_1 t^{n-1} + \cdots + a_n \in \boldsymbol{Z}[t]$ be an irreducible polynomial. The group $G = SL_n$ acts on $X = \{x \in M_n \mid det(tI_n - x) = f(t)\}$ transitively by $(g, x) \mapsto g^{-1}xg$. The stabilizer H of the $\boldsymbol{Q}$-rational point

$$x = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdot & -a_n \\ 1 & 0 & 0 & \cdots & \cdot & -a_{n-1} \\ 0 & 1 & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$$

is the $\boldsymbol{Q}$-anisotropic torus $\mathrm{Ker}(N : R_{K/\boldsymbol{Q}}(\mathbf{G_m}) \to \mathbf{G_m})$, where $K = \boldsymbol{Q}(\alpha)$, $f(\alpha) = 0$, and N is the norm map attached to K/Q.

Then, by Theorem 3.4 and the claim 6.6.1 of [4] which computes $\pi_1(H)_{\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})} = H^{-1}(L/\boldsymbol{Q}, X_*(H))$, if L is the Galois closure of $K/\boldsymbol{Q}$, we have

$$\tau(G, X) = [\mathrm{Coker}(\mathrm{Gal}(L/K)^{ab} \to \mathrm{Gal}(L/\boldsymbol{Q})^{ab})]^{-1},$$

where ab means the abelianization.

For example, if $\mathrm{Gal}(L/\boldsymbol{Q})$ is the symmetric group $S_n (n \geq 3)$, (G, X) has the mean value property.

References

[1] M. V. Borovoi: Non-abelian hypercohomology of a group with coefficients in a crossed module, and Galois cohomology. IAS, Princeton (1991); MPI, Bonn (1992)(preprint).

[2] M. V. Borovoi: Abelianization of the second nonabelian Galois cohomology. Duke Math. J., **72**, 217–239 (1993).

[3] M. V. Borovoi: Abelian Galois cohomology of reductive groups (1995) (preprint).

[4] M. Borovoi and Z. Rudnick: Hardy-Littlewood varieties and semisimple groups. Invent. Math., **119**, 37–66 (1995).

[5] V. I. Chernousov: On the Hasse principle for groups of type E_8. Dokl. Akad. Nauk SSSR, **306**, 25, 1059–1063 (1989).

[6] R. Kottwitz: Tamagawa numbers. Ann. Math., **127**, 3, 629–646 (1988).

[7] J. S. Milne: The points of Shimura varieties modulo a prime of good reduction. The Zeta Functions of Picard Modular Surfaces (eds. R. P. Langlands and D. Ramakrishnan). Montreal: Les Publications CMS, pp. 151–253 (1992). In particular, Appendix B, The cohomology of reductive groups.

[8] M. Morishita and T. Watanabe: A note on the mean value theorem for special homogeneous spaces (to appear in Nagoya Math. J. (1996)).

[9] T. Ono: On the relative theory of Tamagawa numbers. Ann. Math., **82**, 1, 88–111 (1965).

[10] T. Ono: A mean value theorem in adele geometry. J. Math. Soc. Japan, **20**, 275–288 (1968).

[11] T. Ono: Variations on a Theme of Euler, Quadratic forms, Ellitpic curves and Hopf maps. Plenum (1994).

[12] J.-J. Sansuc: Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. J. reine und angew., **327**, 12–80 (1981).

[13] C. L. Siegel: A mean value theorem in geometry of numbers. Ann. Math., **46**, 340–347 (1945).

[14] A. Weil: Sur quelques résultats de Siegel. Summa Brasil Math., **1**, 21–39 (1945–1946), in OEuv. Sci. I, pp. 339–357.

[15] A. Weil: Adeles and Algebraic Groups. Progress in Math., **23**, Birkhäuser, Boston (1982).