Commutation Relations of Differential Operators and Whittaker Functions on $Sp_2(\boldsymbol{R})$*)

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§1. As usual we consider an element in the center of the universal enveloping algebra of Lie algebra of a Lie group G as a differential operator on G. Generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \boldsymbol{R})$ are given in [6].

Put

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_{-i} = {}^tX_i \quad (1 \le i \le 4).$$

Then the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \boldsymbol{R})$ in [6] are

$$\lambda(L_1) = H_1H_1 + H_2H_2 + 6H_1 + 2H_2 + 4X_{-1}X_1 + 8X_{-4}X_4 + 4X_{-3}X_3 + 8X_{-2}X_2,$$

$$\begin{aligned}\lambda(L_2) = {}& 16X_{-4}X_{-4}X_4X_4 + 16X_{-4}X_{-3}X_3X_4 \\ & - 32X_{-4}X_{-2}X_2X_4 + 16X_{-4}X_{-2}X_3X_3 \\ & + 16X_{-4}X_{-1}X_1X_4 + 8X_{-4}H_1H_2X_4 \\ & + 8X_{-4}(H_1 - H_2)X_1X_3 - 16X_{-4}X_1X_1X_2 \\ & + 16X_{-3}X_{-3}X_2X_4 + 16X_{-3}X_{-2}X_2X_3 \\ & + 8X_{-3}X_{-1}(H_1 - H_2)X_4 + 4X_{-3}H_2H_2X_3 \\ & + 8X_{-3}(H_1 + H_2)X_1X_2 + 16X_{-2}X_{-2}X_2X_2 \\ & - 16X_{-2}X_{-1}X_{-1}X_4 + 8X_{-2}X_{-1}(H_1 + H_2)X_3 \\ & + 16X_{-2}X_{-1}X_1X_2 - 8X_{-2}H_1H_2X_2 \\ & + 4X_{-1}H_1H_1X_1 + H_1H_1H_2H_2 - 16X_{-4}H_1X_4 \\ & + 32X_{-4}H_2X_4 + 32X_{-4}X_1X_3 + 32X_{-3}X_{-1}X_4 \\ & - 8X_{-3}H_1X_3 + 16X_{-3}X_1X_2 + 16X_{-2}X_{-1}X_3 \\ & - 16X_{-2}(H_1 + H_2)X_2 + 24X_{-1}H_1X_1 \\ & + 2H_1H_1H_2 + 6H_1H_2H_2 - 48X_{-4}X_4 \\ & - 24X_{-3}X_3 - 48X_{-2}X_2 + 24X_{-1}X_1 - 2H_1H_1 \\ & + 12H_1H_2 + 6H_2H_2 - 12H_1 + 12H_2.\end{aligned}$$

We can find the generators of the centers of the universal enveloping algebras of the Lie algebras of classical groups by [6], [3]. The generators of the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}(4, \boldsymbol{R})$ are f_2, f_4, f_6 in [3, §13, no. 4, (VI), p. 189]. The polynomial functions f_2, f_4, f_6 on $\mathfrak{g}$ are the coefficients of the characteristic polynomial of the identity representation of $\mathfrak{g}$. Identifying the dual of $\mathfrak{g}$ with $\mathfrak{g}$ and applying a symmetrizer Λ such that

$$\Lambda(X_1X_2\ldots X_n) = \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)}X_{\sigma(2)}\ldots X_{\sigma(n)} \quad (X_i \in \mathfrak{g})$$

on the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ to f_2, f_4, f_6, we get the generators $\beta_2 = \Lambda(f_2)$, $\beta_3 = \Lambda(f_4)$, $\beta_4 = \Lambda(f_6)$ of the center of $U(\mathfrak{g})$.

§2. We define the Weil representation r_n of $Sp_2(\boldsymbol{R})$ on $\mathcal{S}(V_n \times V_n)$, $V = V_n = M_{n,2}(\boldsymbol{R})$ by putting

$$r_n\begin{pmatrix} E & X \\ 0 & E \end{pmatrix} f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \exp(2\pi i \operatorname{tr}(X\,{}^tX_1X_2))\, f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

$$r_n\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = (\det A)^n f\begin{pmatrix} X_1A \\ X_2A \end{pmatrix},$$

$$r_n\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \int_{V_n}\int_{V_n} \exp(2\pi i \operatorname{tr}({}^tY_1{}^tX_2 + {}^tY_2X_1))\, f\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} dY_1dY_2$$

for $f \in \mathcal{S}(V_n \times V_n)$, $X = {}^tX \in M_{2,2}(\boldsymbol{R})$, $A \in M_{2,2}(\boldsymbol{R})$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X_1 \in V_n$, $X_2 \in V_n$.

Put $G_1 = SL(2, \boldsymbol{R})$, $G_3 = SL(4, \boldsymbol{R})$. Then we can define representations ρ_2, ρ_3 of $G_1 \times G_1$, G_3 on $\mathcal{S}(V_2 \times V_2)$, $\mathcal{S}(V_3 \times V_3)$ in the following manner. First we define linear mappings σ_1, σ_3 by

$$\sigma_1(X) = \begin{pmatrix} a & d \\ b & -c \end{pmatrix} \text{ for } X = {}^t(a\ b\ c\ d) \in M_{4,1}(\boldsymbol{R})$$

and

*) Dedicated to Professor Hideo Shimizu on his sixtieth birthday.

$$\sigma_3(X)=\begin{pmatrix}0 & a & b & c\\ -a & 0 & f & -e\\ -b & -f & 0 & d\\ -c & e & -d & 0\end{pmatrix}$$

$$\text{for } X=\begin{pmatrix}a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}\in M_{6,1}(\boldsymbol{R}).$$

Then $(g, h)\in G_1\times G_1$ acts on $V_2\times V_2$ by

$$\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{(g,h)}=\left(\sigma_1^{-1}\left({}^t g\left(\sigma_1\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}\right)h\right),\ \sigma_1^{-1}\left({}^t g\left(\sigma_1\begin{pmatrix}a'\\ b'\\ c'\\ d'\end{pmatrix}\right)h\right)\right)$$

for

$$\begin{pmatrix}X_1\\ X_2\end{pmatrix}=\begin{pmatrix}a & a'\\ b & b'\\ c & c'\\ d & d'\end{pmatrix}\in M_{4,2}(\boldsymbol{R})=V_2\times V_2,$$

and $g\in G_3$ acts on $V_3\times V_3$ by

$$\begin{pmatrix}X_1\\ X_2\end{pmatrix}^{g}=\left(\sigma_3^{-1}\left({}^t g\left(\sigma_3\begin{pmatrix}a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}\right)g\right),\ \sigma_3^{-1}\left({}^t g\left(\sigma_3\begin{pmatrix}a'\\ b'\\ c'\\ d'\\ e'\\ f'\end{pmatrix}\right)g\right)\right)$$

for

$$\begin{pmatrix}X_1\\ X_2\end{pmatrix}=\begin{pmatrix}a & a'\\ b & b'\\ c & c'\\ d & d'\\ e & e'\\ f & f'\end{pmatrix}\in M_{6,2}(\boldsymbol{R})=V_3\times V_3.$$

Put

$$\rho_2(g)f(X)=f(X^g)$$

for $f\in\mathcal{S}(V_2\times V_2)$, $g\in G_1\times G_1$, and put

$$\rho_3(g)f(X)=f(X^g)$$

for $f\in\mathcal{S}(V_3\times V_3)$, $g\in G_3$. Then the representations r_2, r_3, ρ_2, ρ_3 induce the representations (differential representations) of the centers of the universal enveloping algebras of $\mathfrak{sp}(2,\boldsymbol{R})$, $\mathfrak{sp}(2,\boldsymbol{R})$, $\mathfrak{sl}(2,\boldsymbol{R})\oplus\mathfrak{sl}(2,\boldsymbol{R})$, $\mathfrak{sl}(4,\boldsymbol{R})$ which we denote by the same letters r_2, r_3, ρ_2, ρ_3. Let γ be

$$2\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)+\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then by using a computer machine for instance we can check

Theorem 1.

$$\rho_3(\beta_2)=-\frac{1}{128}r_3(\lambda(L_1)),\ \rho_3(\beta_3)=0,$$

$$\rho_3(\beta_4)=-\frac{3}{8192}r_3(\lambda(L_2))+\frac{1}{2048}r_3(\lambda(L_1)),$$

$$r_2(\lambda(L_1))=\rho_2(\gamma, 0)+\rho_2(0,\gamma)-8,$$

$$r_2(\lambda(L_2))=\rho_2(\gamma, 0)\rho_2(0,\gamma)-2\rho_2(\gamma, 0)-2\rho_2(0,\gamma)+16.$$

§3. By using Theorem 1 we can construct standard Whittaker functions on $Sp_2(\boldsymbol{R})$. First we consider same theta functions $\Theta(g, z_1, z_2)$ as in [8] attached to the Weil representation r_2 and define a lift

$$F(g)=F_{\varphi_1,\varphi_2}(g)=\int_{\Gamma\backslash H}\int_{\Gamma\backslash H}\Theta(g, z_1, z_2)\varphi_1(z_1)\varphi_2(z_2)\,d_0z_1d_0z_2$$

where φ_1, φ_2 are Mass wave forms on the upper half plane H. We consider the case where the level is one and $\Gamma=SL_2(\boldsymbol{Z})$. Define a character Ψ_0 by

$$\Psi_0\left(\begin{pmatrix}1 & x_0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -x_0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & x_1 & x_2\\ 0 & 1 & x_2 & x_3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right)=\exp(2\pi i(x_0+x_3))$$

of the unipotent radical N of a Borel subgroup of $Sp_2(\boldsymbol{R})$. Considering

$$\int_{N\cap Sp_2(\boldsymbol{Z})\backslash N}F(ng)\Psi_0(n)\,dn,$$

we get a following Whittaker function

$$W_{\nu_1,\nu_2}\left(\begin{pmatrix}1 & x_0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -x_0 & 1\end{pmatrix}\begin{pmatrix}y_1 & 0 & x_1/y_1 & x_2/y_2\\ 0 & y_2 & x_2/y_1 & x_3/y_2\\ 0 & 0 & 1/y_1 & 0\\ 0 & 0 & 0 & 1/y_2\end{pmatrix}k\right)$$

$$=\exp(2\pi i(x_0+x_3))\int_0^\infty\int_0^\infty v_1^{-1}v_2^{-1}y_1^2y_2K_{\nu_1}(2\pi v_1)K_{\nu_2}(2\pi v_2)$$

$$\exp(-\pi y_1^2/v_1v_2 - \pi v_1v_2/y_2^2 - \pi v_1y_2^2/v_2 - \pi v_2y_2^2/v_1)\,dv_1dv_2$$

for $k \in SO(4) \cap Sp_2(\boldsymbol{R})$ with the modified Bessel function K_ν. (See [5].) The latter part of Theorem 1 or a direct computation implies

Theorem 2. *For* $\nu_1, \nu_2 \in \sqrt{-1}\boldsymbol{R}$

$$\lambda(L_1)W_{\nu_1,\nu_2} = 4(\lambda_1 + \lambda_2 - 2)W_{\nu_1,\nu_2},$$
$$\lambda(L_2)W_{\nu_1,\nu_2} = 8(2\lambda_1\lambda_2 - \lambda_1 - \lambda_2 + 2)W_{\nu_1,\nu_2}$$

with $\lambda_1 = \nu_1^2 - 1/4$, $\lambda_2 = \nu_2^2 - 1/4$.

When we put $y_1 = u_1u_2$, $y_2 = u_2$, we can easily see that W_{ν_1,ν_2} is rapidly decreasing as u_1 or u_2 tends to infinity. The uniqueness of the function having these properties is well known and so we have got an explicit formula of the class one Whittaker function. By Theorem 2 and

$$\int_0^\infty \exp\left(-\frac{y}{2} - \frac{z^2+w^2}{2y}\right)K_\nu\left(\frac{zw}{y}\right)\frac{dy}{y} = 2K_\nu(z)K_\nu(w),$$

we can easily show

Theorem 3. *For* $\nu_1, \nu_2, \nu \in \sqrt{-1}\,\boldsymbol{R}$ *and sufficiently large* $\mathrm{Re}(s)$

$$\int_0^\infty\int_0^\infty W_{\nu_1,\nu_2}\left(\begin{pmatrix} \sqrt{y_1} & 0 & 0 & 0 \\ 0 & \sqrt{y_2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{y_1} & 0 \\ 0 & 0 & 0 & 1/\sqrt{y_2} \end{pmatrix}\right) \sqrt{y_2}K_\nu(2\pi y_2)y_1^{s-2}y_2^{-2}dy_1dy_2$$
$$= 2^{-5}\pi^s\Gamma(s)^{-1}$$
$$\pi^{-2s}\Gamma\left(\frac{s+\nu_1+\nu}{2}\right)\Gamma\left(\frac{s-\nu_1+\nu}{2}\right)\Gamma\left(\frac{s+\nu_1-\nu}{2}\right)\Gamma\left(\frac{s-\nu_1-\nu}{2}\right)$$
$$\pi^{-2s}\Gamma\left(\frac{s+\nu_2+\nu}{2}\right)\Gamma\left(\frac{s-\nu_2+\nu}{2}\right)\Gamma\left(\frac{s+\nu_2-\nu}{2}\right)\Gamma\left(\frac{s-\nu_2-\nu}{2}\right)$$

We note that the right hand side of the above equality equals the Γ-factor of the product L-function associated with a Siegel wave form belonging to the same eigenvalues as in Theorem 2 and a Maass wave form belonging to the eigenvalue $4\nu^2 - 1$ of the Casimir operator γ defined before Theorem 1.

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