14. *Determination of All Quaternion Octic CM-fields with Ideal Class Groups of Exponents 2 Abridged Version*

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In [9] the authors set to determine the non-abelian normal CM-fields with class number one. Since they have even relative class numbers, they got rid of quaternion octic CM-fields. Here, a quaternion octic field is a normal number fields of degree 8 whose Galois group is the quaternion group $\mathbf{G} = \{\pm 1, \pm i, \pm j, \pm k\}$ with $ij = k$, $jk = i$, $ki = j$ and $i^2 = j^2 = k^2 = -1$. Then, in [8] the first author determined the only quaternion octic CM-fields with class number 2. Here, we delineate the proof of the following result proved in [10] that generalizes this previous result:

Theorem. *There are exactly* 2 *quaternion octic* CM-*fields with ideals class groups of exponents* 2. *Namely, the following two pure quaternion number fields*:

$$Q(\sqrt{-(2+\sqrt{2})(3+\sqrt{6})})$$

with discriminant $2^{24}3^6$ *and class number* 2, *and*

$$Q(\sqrt{-(5+\sqrt{5})(5+\sqrt{21})(21+\sqrt{105})})$$

with discriminant $3^6 5^6 7^6$ *and class number* 8.

1. Analytic lower bounds for relative class numbers and maximal real subfields of quaternion octic CM-fields with ideal class groups of exponents 2. Here we show that under the assumption of a suitable hypothesis (H) we can set lower bounds on relative class numbers of quaternion octic CM-fields. Let us remind the reader that a number field N is called a CM-field if it is a totally imaginary number field that is a quadratic extension of a totally real subfield K. In that situation, one can prove that the class number h_K of K divides that h_N of N, and the relative class number h_N^- of N is defined by means of $h_N^- = h_N / h_K$ (see [11, Theorem 4.10]). Note h_N^- divides h_N.

Proposition 1. (a). (See [5, Theorems 1 and 2(a)]) *Let N be a quaternion octic* CM-*field such that the Dedekind zeta function of its real bicyclic biquadratic subfield K satisfies*

$$\text{(H)} \qquad \zeta_K\left(1 - \frac{2}{\log(D_N)}\right) \leq 0.$$

Then, we have the following lower bound for the relative class number h_N^- of N:

$$\text{(1)} \qquad h_N^- \geq \left(1 - \frac{8\pi e^{1/4}}{D_N^{1/8}}\right) \frac{1}{4e\pi^4} \frac{1}{\operatorname{Res}_{s=1}(\zeta_K)} \frac{\sqrt{D_N/D_K}}{\log(D_N)}.$$

Moreover, the hypothesis (H) *is satisfied provided that we have*

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(2) $$h_N^- \leq \frac{1}{16e}\sqrt{\frac{D_N}{D_K^2}}.$$

(b). (See [6]) *Set* $c = 2 + \gamma - \log(4\pi)$, *where* $\gamma = 0.577215\cdots$ *is the Euler's constant, so that we have* $3c \leq 0.14$. *Then,*

(3) $$\mathrm{Res}_{s=1}(\zeta_K) \leq \frac{1}{216}(\log(D_K) + 3c)^3.$$

In order to show that the hypothesis (H) is satisfied whenever N is a quaternion octic CM-field with ideal class group of exponent 2, we would like to show that h_N^- is not too large, i.e. is such that (2) is satisfied. Hence, we would like to be able to compute the 2-rank of the ideal class group of N. It is not hard to see that the ambiguous class number formula (see [1] or [3]) provides us with the determination of the 2-rank of the ideal class group of any CM-field N such that its maximal totally real subfield K has odd class number. Hence, we would like to prove that the real bicyclic biquadratic subfield K of any quaternion CM-field with ideal class group of exponent 2 has odd number.

This task is accomplished by use of Fröhlich's description [2] of quaternion octic fields and delicate examination of ideal characters of quadratic subfields paying respect to difficulty coming from unit groups:

Lemma A. *Let* k *be a real quadratic field,* ε^+ *the totally positive fundamental unit and* N/k *a cyclic quartic extension. Assume that prime numbers* $p_1, p_2, \ldots, p_l$ *remain inert in* k/Q *and completely ramify in* N/k*. Then the 4-rank of the class group of* N *is non-zero if* $l \geqslant 3$ *or* ε^+ *is norm-residue at* (p_1) *in* N/k *with* $l \geqslant 2$.

In fact, we determine possible (necessary) forms of quaternion octic CM-*fields whose class group have no elements of order* 4:

Theorem 2. *Let* N *be a quaternion octic* CM-*field and suppose that the 4-rank of the ideal class group of* N *is equal to zero. Let* K *be the real bicyclic biquadratic subfield of* N. *Let* k_i, $1 \leq i \leq 3$ *be the three real quadratic subfields of* K. *Let* T *be the number of ramified prime numbers in* K *and let* $t_{K/Q}$ *be the number of prime ideals of* K *that are ramified in* K/Q. *Finally, let* Q_K *be the unit index* $(U_K : U_{k_1}U_{k_2}U_{k_3})$, *so that we have the following class numbers relation:*

$$h_K = \frac{Q_K}{4} h_{k_1} h_{k_2} h_{k_3}.$$

Then, K *is one of the following eight forms:*

1. $K = Q(\sqrt{2}, \sqrt{q})$ *with* $q \equiv 3 \pmod 8$. *Then,* $t_{K/Q} = T = 2$ *and* $Q_K = 4$.
2. $K = Q(\sqrt{2}, \sqrt{qr})$ *with* $q \equiv r \equiv 3 \pmod 8$. *Then,* $t_{K/Q} = 4$, $T = 3$ *and* $Q_K = 2$.
3. $K = Q(\sqrt{p}, \sqrt{2r})$ *with* $p \equiv 5 \pmod 8$, $r \equiv 3 \pmod 8$ *and* $(p/r) = -1$. *Then,* $t_{K/Q} = 4$, $T = 3$ *and* $Q_K = 2$.
4. $K = Q(\sqrt{p}, \sqrt{qr})$ *with* $p \equiv 1 \pmod 4$, $q \equiv r \equiv 3 \pmod 4$ *and* $(p/q) = (p/r) = -1$. *Then,* $t_{K/Q} = 4$, $T = 3$ *and* $Q_K = 2$.
5. $K = Q(\sqrt{2q}, \sqrt{qr})$ *with* $q \equiv 7 \pmod 8$, $r \equiv 3 \pmod 8$ *and* $(r/q) = -1$. *Then,* $t_{K/Q} = T = 3$ *and* $Q_K = 4$.

6. $K = Q(\sqrt{pq}, \sqrt{qr})$ *with* $p \equiv q \equiv r \equiv 3 \pmod 4$ *and* $(q/p) = (r/q) = (p/r) = -1$. *Then*, $t_{K/Q} = T = 3$ *and* $Q_K = 4$.
7. $K = Q(\sqrt{2}, \sqrt{q})$ *with* $q \equiv 1 \pmod 8$ *and* $(2/q)_4(q/2)_4 = -1$. *Then*, $t_{K/Q} = 4$, $T = 2$ *and* $Q_K = 2$.
8. $K = Q(\sqrt{p}, \sqrt{q})$ *with* $p \equiv q \equiv 1 \pmod 4$, $(p/q) = 1$ *and* $(p/q)_4(q/p)_4 = -1$. *Then*, $t_{K/Q} = 4$, $T = 2$ *and* $Q_K = 2$.

In each of these eight cases, the class number of K *is odd. Let* U_K *and* U_K^+ *be the group of units of* K *and the group of totally positive units of* K. *Then, in each of these eight cases we have* $(U_K^+ : U_K^2) = 2$. *Hence*, $(U_K \cap N_{N/K}(N^+) : U_K^2) = 2^\rho$ *is equal to* 1 *or* 2. *Moreover, except possibly in cases* 5 *and* 6, *we have* $\rho = 0$. *Hence, in each of these eight cases we get* $t_{K/Q} + \rho \leq 4$.

2. Scheme of the proof of the theorem. Now our strategy is as follows.

First, using the ambiguous class number formula, we show in Lemmas **B** and **C** that if a quaternion octic CM-field has an ideal class group of exponent 2 then (2) is satisfied, except for a finite number of K's for which we use in Lemma **D** a trick that shows that the hypothesis (H) is satisfied.

Second, using (1) and (3) we get the upper bound $D_K \leq 25 \cdot 10^6$ on the discriminants of real bicyclic biquadratic subfields of quaternion octic CM-fields with ideal class groups of exponents 2. Then, we give a short list of real bicyclic biquadratic number fields K that can be subfields of quaternion octic CM-fields with ideal class groups of exponents 2 (see Lemma **E**). Then, for each possible K we get a finite list of possible values for discriminants of quaternion octic CM-fields with ideal class groups of exponent 2 containing this number field K (see Lemma **F**).

Third, using the method developed in [7], we compute the relative class numbers of the quaternion octic CM-fields of discriminants belonging to this list.

For any quaternion octic number field N with bicyclic biquadratic subfield K, we can find a pure quaternion octic number field N_0 and a discriminant Δ of a quadratic number field such that $N \subset N_0(\sqrt{\Delta})$. The discriminant D_N of N is then equal to $D_{N_0}\Delta^4$, and the discriminant D_{N_0} of N_0 is $D_{N_0} = 16D_K^3$ if 2 has ramification index equal to 2 in K/Q, and $D_{N_0} = D_K^3$ otherwise.

Lemma B. *If* N *is a quaternion octic* CM-*field with ideal class group of exponent* 2, *then* $h_K = 1$ *and* $h_N^- \leq 2^{4m+3}$ *where* m *is the number of distinct prime divisors of* Δ. *More precisely, the* 2-*rank of the ideal class group of* N *is* $t_{N/K} - 1 + \rho$ *where* $t_{N/K}$ *is the number of prime ideals of* K *that ramify in the quadratic extension* N/K, *and* $\rho \in \{0, 1\}$ *as in Theorem* 2.

For $m \geq 0$, set $\Delta_0 = 1$ and $\Delta_m = l_1 \cdots l_m$, $3 = l_1 < 4 = l_2 < 5 = l_3 < \cdots < l_m$ where the l_i's, $i \geq 3$ is the increasing sequence of odd primes greater than 3. Hence, with m being as in Lemma **B**, we have $D_N \geq D_K^3\Delta_m^4$.

Lemma C. *If the ideal class group of a quaternion octic* CM-*field* N *is of exponent* 2, *then the hypothesis* (H) *of Proposition* 1 *is satisfied provided that we have* $D_K \geq 382617$.

Proof. Noticing that $h_N^- \leq 8 \cdot 4^{2m}$ (see Lemma **B**) and $\sqrt{D_N/D_K^2} \geq \Delta_m^2$

$\sqrt{D_K}$, it suffices to show that (2) is satisfied, hence it suffices to show that we have

$$(4) \qquad \left(\frac{\Delta_m}{4^m}\right)^2 = \left(\frac{l_1}{4}\frac{l_2}{4}\cdots\frac{l_m}{4}\right)^2 \geq \frac{128e}{\sqrt{D_K}}.$$

Since the left hand side of (4) is greater than or equal to $(3/4)^2$, then (4) is satisfied if $D_K \geq 382617$.

Using the fact that the Dedekind zeta function of a bicyclic biquadratic number field is the product of the Riemann zeta function and of the three L-functions associated to the three characters of the three quadratic subfields of K, we have the following result that will enable us to show that the hypothesis (H) is satisfied when we have $D_K \leq 382616$.

Lemma D. *Let k be a real quadratic field of conductor f and quadratic character χ. Then, the Dedekind zeta function of k is negative on $]0,1[$ provided that $S(n) = \sum_{a=1}^{n}\sum_{b=1}^{a}\chi(n)$ $S(n)$* satisfies $\geqslant 0$, $1 \leq n \leq f$.

3. Upper bounds on the discriminants of the bicyclic biquadratic real subfields of quaternion octic CM-fields with ideal class groups of exponents 2. Let us assume that K is a quartic subfieid of a quaternion octic CM-field N with ideal class group of exponent 2 such that the hypothesis (H) is satisfied. Then, since $D_N \geq D_K^3 \Delta_m^4$ and $h_N^- \leq 2^{4m+3}$, (1) and (3) we have:

$$(5) \qquad f_K(m) := \left(1 - \frac{8\pi e^{1/4}}{D_K^{3/8}}\right)\frac{D_K\Delta_m^2 16^{-m}}{(\log(D_K)+0.14)^3\log(D_K^3\Delta_m^4)} \leq \frac{4e\pi^4}{27}.$$

Now, one can easily see that we have $f_K(m) \geq f_K(2)$, $m \geq 0$. Hence (5) implies

$$(6) \qquad \left(1 - \frac{8\pi e^{1/4}}{D_K^{3/8}}\right)\frac{D_K}{(\log(D_K)+0.14)^3\log(12^4 D_K^3)} \leq \frac{64e\pi^4}{243}.$$

One can easily check that (6) implies

$$D_K \leq 25\cdot 10^6$$

Moreover, instead of using (3), for a fixed K that satisfies hypothesis (H) let us use (1). We get a more restrictive inequality than (6), namely:

$$(7) \qquad \left(1 - \frac{8\pi e^{1/4}}{D_K^{3/8}}\right)\frac{D_K}{\log(12^4 D_K^3)} \leq \frac{512e\pi^4}{9}\mathrm{Res}_{s=1}(\zeta_K).$$

This inequality (7) will enable us to get rid of most of the number fields K that satisfy (6).

Moreover, if we assume that 2 has ramification index 2 in K, or if 2 is totally ramified in K, then we can state much more satisfactory inequalities.

4. Upper bounds on the discriminants of the quaternion octic CM-fields with ideal class groups of exponents 2. Now, for each field K we use (5) to put an upper bound m_{max} on m, and then we use (6) with $h_N^- = 2^{4m_{max}+3}$ to put an upper bound on D_N. Finally, using this upper bound on D_N, for each K and each D_N we compute the exact value of $t_{N/K}$ and use the upper bound $h_N^- \leq 2^{t_{N/K}-1}$ in (1), i.e. we use

$$(8) \qquad \left(1 - \frac{8\pi e^{1/4}}{(D_N)^{1/8}}\right)\frac{\sqrt{D_N/D_K}}{\log(D_N)} \leq 2e\pi^4 2^{t_{N/K}+\rho}\mathrm{Res}_{s=1}(\zeta_K)$$

to get rid of many number fields N.

5. Full proof for case 4 of Theorem 2. We explain on one of the eight possible forms for K how we get upper bounds on discriminants of quaternion octic CM-fields with ideal class groups of exponents 2. Hence, we assume that N be a quaternion octic CM-field that is a quadratic extension of the real bicyclic biquadratic field $K_{(p,qr)} = Q(\sqrt{p}, \sqrt{qr})$, with $p \equiv 1 \pmod 4$ and $q \equiv r \equiv 3 \pmod 4$ three distinct primes such that $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = -1$. Then, $\rho = 0$ and $K_{(p,qr)}$ has odd class number, so that the 2-rank of the ideal class group of N is $t_{N/K_{(p,qr)}} - 1$. Moreover, $D_{K_{(p,qr)}} = (pqr)^2$ and $D_N = (pqr)^6 \Delta^4$ where $\Delta \geq 1$ is prime to pqr and is a square-free or four times a square-free positive integer. Moreover, we have

$$\begin{aligned} pqr &= 5\cdot 3\cdot 7 = 105 \\ &= 5\cdot 3\cdot 23 = 345 \\ &= 17\cdot 3\cdot 7 = 357 \\ &= 17\cdot 3\cdot 11 = 561 \end{aligned}$$

or $pqr \geq 5 \cdot 3 \cdot 43 = 645$ which implies $D_{K_{(p,qr)}} \geq 382617$. Using Lemma D for the four previous values of pqr, we thus get that the hypothesis (H) is satisfied whenever $K_{(p,qr)}$ is a quartic subfield of a quaternion octic CM-field with ideal class group of exponent 2. Now, we lower our previous upper bound on D_K. Indeed, for the 65 number fields $K_{(p,qr)}$'s such that $D_{K_{(p,qr)}} \leq 25 \cdot 10^6$, we use (7) instead of (6). We thus get that only 8 out of these 65 quartic number fields could be quartic subfields of quaternion CM-fields with ideal class groups of exponents 2, i.e., we have proved:

Lemma E. *If $K_{(p,qr)}$ is the quartic subfield of a quaternion octic* CM*-field with ideal class group of exponent* 2, *then* $(p, q, r) \in \{(5,3,7) ; (5,3,23) ; (17,3,7), ; (17,3,11) ; (5,3,47) ; (5,7,23) ; (41,3,7) ; (41,3,11)\}$.

We point out that these eight real quartic fields $K_{(p,qr)}$'s have class number one. Let us point out that here we have $\rho = 0$. Now, using (8), we get:

Lemma F. *If N is a quaternion octic* CM*-field with ideal class group of exponent* 2 *that is a quadratic extension of some $K_{(p,qr)}$, then we have*:

(p,qr)	$D_N \in$	*2-rank of the class group of N*
(5,21)	$\{(5\cdot3\cdot7)^6, (5\cdot3\cdot7)^6 4^4, (5\cdot3\cdot7)^6 8^4\}$	3,5,5
(5,69)	$\{(5\cdot3\cdot23)^6\}$	3
(17,33)	$\{(17\cdot3\cdot11)^6, (17\cdot3\cdot11)^6 4^4, (17\cdot3\cdot11)^6 8^4\}$	3,7,7.

Now, using [7] we compute the relative class numbers of the 9 possible CM-fields N whose discriminants are given in Lemma F. We get the following table:

$$N = N_{(5,3\cdot7,1)} = Q\left(\sqrt{-\frac{5+\sqrt{5}}{2}(21+2\sqrt{105})\frac{5+\sqrt{21}}{2}}\right) \qquad h_N^- = 2^3$$

$$N = N_{(5,3\cdot7,4)} = Q\left(\sqrt{-4\frac{5+\sqrt{5}}{2}(21+2\sqrt{105})}\right) \qquad h_N^- = 2^5\cdot 3^2$$

$$N = N_{(5,3\cdot7,8)} = Q\left(\sqrt{-8\frac{5+\sqrt{5}}{2}(21+2\sqrt{105})\frac{5+\sqrt{21}}{2}}\right) \qquad h_N^- = 2^5\cdot 5^2$$

$$N' = N'_{(5,3\cdot 7,8)} = Q\left(\sqrt{-8\frac{5+\sqrt{5}}{2}(21+2\sqrt{105})}\right) \qquad h^-_{N'} = 2^5\cdot 5^2$$

$$N = N_{(5,3\cdot 23,1)} = Q\left(\sqrt{-\frac{5+\sqrt{5}}{2}(483+26\sqrt{345})}\right) \qquad h^-_N = 2^3\cdot 3^2$$

$$N = N_{(17,3\cdot 11,1)} = Q(\sqrt{-(17+4\sqrt{17})(2937+124\sqrt{561})(23+4\sqrt{33})}) \qquad h^-_N = 2^3\cdot 3^2$$

$$N = N_{(17,3\cdot 11,4)} = Q(\sqrt{-4(17+4\sqrt{17})(2937+124\sqrt{561})}) \qquad h^-_N = 2^9\cdot 3^2$$

$$N = N_{(17,3\cdot 11,8)} = Q(\sqrt{-8(17+4\sqrt{17})(2937+124\sqrt{561})(23+4\sqrt{33})}) \qquad h^-_N = 2^7\cdot 13^2$$

$$N' = N'_{(17,3\cdot 11,8)} = Q(\sqrt{-8(17+4\sqrt{17})(2937+124\sqrt{561})}) \qquad h^-_{N'} = 2^9\cdot 7^2.$$

Since the real bicyclic biquadratic number field $K_{(5,21)} = Q(\sqrt{5}, \sqrt{21})$ has class number one, we have proved that there exists exactly one quaternion CM-field N containing some $K_{(p,qr)}$ that has an ideal class group of exponent 2, namely the pure quaternion field

$$N_{(5,3\cdot 7,1)} = Q\left(\sqrt{-\frac{5+\sqrt{5}}{2}\frac{5+\sqrt{21}}{2}(21+2\sqrt{105})}\right).$$

Its ideal class group is isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^3$.

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