# 13. *Associated Varieties and Gelfand-Kirillov Dimensions for the Discrete Series of a Semisimple Lie Group*

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(Communicated by Kiyosi ITÔ, M. J. A., Feb. 14, 1994)

**1. Introduction.** Let $G$ be a connected semisimple Lie group with finite center, and $K$ be a maximal compact subgroup of $G$. The corresponding complexified Lie algebras are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. We assume Harish-Chandra's rank condition $\mathrm{rank}\, G = \mathrm{rank}\, K$, which is necessary and sufficient for $G$ to have a non-empty set of discrete series, or of square-integrable irreducible unitary representations of $G$.

In this paper, we describe the associated varieties of Harish-Chandra $(\mathfrak{g}, K)$-modules of discrete series, by an elementary and direct method based on [3]. The description is as in

**Theorem 1.** *If $H_\Lambda$ is the $(\mathfrak{g}, K)$-module of discrete series with Harish-Chanda parameter $\Lambda = \lambda + \rho_c - \rho_n$ (see §3), then its associated variety $\mathcal{V}(H_\Lambda) \subset \mathfrak{g}$ (see §2) coincides with the nilpotent cone $K_C\mathfrak{p}_-$, which is equal to $\mathrm{Ad}(K)\mathfrak{p}_-$. Here $K_C$ denotes the analytic subgroup of adjoint group $G_C := \mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$, with Lie algebra $\mathfrak{k}$, and $\mathfrak{p}_- = \sum_{\beta\in\Delta_n^-}\mathfrak{g}_\beta$ is the sum of root subspaces $\mathfrak{g}_\beta$ of $\mathfrak{g}$ corresponding to the noncompact roots $\beta$ such that $(\Lambda, \beta) < 0$.*

We further give in Theorem 4 an explicit formula for the Gelfand-Kirillov dimensions $d(H_\Lambda)\ \dim \mathcal{V}(H_\Lambda)$ of discrete series in the case of unitary groups $G = SU(p, q)$, by specifying the unique nilpotent $G_C$-orbits in $\mathfrak{g}$ which intersect $\mathfrak{p}_-$ densely. Note that this important invariant $d(H_\Lambda)$ coincides with the degree of Hilbert polynomial of $H_\Lambda$.

We know that Theorem 1 can be deduced from deep results in [1, III] and [4] by passing to $D$-module via Beilinson-Bernstein correspondence. However, the associated variety is an object attached directly to each finitely generated $U(\mathfrak{g})$-module. From this reason, we give here a direct path to the theorem avoiding the above detour by $D$-module. Our proof of Theorem 1 is simple in the sense that it uses only some basic results of [3] on the realization of $H_\Lambda$ as the kernel space of differential operator $\mathcal{D}_\lambda$ on $G/K$ of gradient-type. Nevertheless, this method gives us new conclusions also (Theorem 3). For instance, we find that the associated variety of discrete series can be expressed in terms of the symbol mapping of $\mathcal{D}_\lambda$.

**2. Associated varieties for $U(\mathfrak{g})$-modules.** Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and $(U_k(\mathfrak{g}))_{k=0,1,..}$ be the natural increasing filtration of $U(\mathfrak{g})$, with $U_k(\mathfrak{g})$ the subspace of $U(\mathfrak{g})$ generated by elements $X^m (0 \leq m \leq k, X \in \mathfrak{g})$. We identify the associated graded ring $\mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_{k\geq 0} U_k(\mathfrak{g}) / U_{k-1}(\mathfrak{g})$ $(U_{-1}(\mathfrak{g}) := (0))$ with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k\geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. Here $S^k(\mathfrak{g})$ denotes the homogeneous component of

$S(\mathfrak{g})$ of degree $k$.

For a finitely generated $U(\mathfrak{g})$-module $H$, take a finite-dimensional subspace $H_0$ of $H$ such that $H = U(\mathfrak{g})H_0$, and set $H_k := U_k(\mathfrak{g})H_0 (k = 1,2,\dots)$. Then $(H_k)_k$ gives an increasing filtration of $H$, and corresponding one gets a finitely generated, graded $S(\mathfrak{g})$-module $M := \oplus_{k\geq 0} M_k$ with $M_k = H_k/H_{k-1}$.

The annihilator ideal $\mathrm{Ann}_{S(\mathfrak{g})}M := \{D \in S(\mathfrak{g}) \mid Dv = 0 \ (\forall\, v \in M)\}$ of $M$ in $S(\mathfrak{g})$ defines an algebraic cone in $\mathfrak{g}$:

(2.1) $$\mathscr{V}(H) := \{X \in \mathfrak{g} \mid f(X) = 0 \ (\forall f \in \mathrm{Ann}_{S(\mathfrak{g})} M)\},$$

which is independent of the choice of a subspace $H_0$. Here $S(\mathfrak{g})$ is viewed as the polynomial ring over $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$. The variety $\mathscr{V}(H)$ and its dimension $d(H) := \dim \mathscr{V}(H)$ are called respectively the *associated variety* and the *Gelfand-Kirillov dimension* of $H$ (cf. [5, 6, 8]).

**3. Discrete series for $G$.** We now fix some notation on the discrete series representations of $G$ (cf. [2]). Take a compact Cartan subgroup $T$ of $G$ contained in $K$. Let $\Delta$ be the root system of $\mathfrak{g}$ with respect to the complexified Lie algebra $\mathfrak{k}$ of $T$. The totality of compact (resp. noncompact) roots in $\Delta$ will be denoted by $\Delta_c$ (resp. $\Delta_n$). Fix once and for all a positive system $\Delta_c^+$ of $\Delta_c$. Let $\Xi$ be the set of $\Delta_c^+$-dominant, $\Delta$-regular linear forms $\Lambda$ on $\mathfrak{k}$ such that $\Lambda + \rho$ is $T$-integral through the exponential map. Here $\rho := (1/2)\sum_{\alpha\in\Delta^+}\alpha$ with $\Delta_+ = \{\alpha \in \Delta \mid (\Lambda, \alpha) > 0\}$.

By Harish-Chandra, there exists a natural bijective correspondence, say $\Lambda \to \pi_\Lambda$, from $\Xi$ onto the set of (equivalence classes) of discrete series representations of $G$. By taking the $K$-finite part for $\pi_\Lambda$, one gets an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module, which we denote by $H_\Lambda$ from now on.

For a $\Delta_c^+$-dominant, $T$-integral linear form $\mu \in \mathfrak{k}^*$, let $(\tau_\mu, V_\mu)$ denote the irreducible $K$-module with highest weight $\mu$. Set for a $\Lambda \in \Xi$,

(3.1) $$\lambda := \Lambda - \rho_c + \rho_n, \text{ with } \rho_c := (1/2)\cdot\sum_{\alpha\in\Delta_c^+}\alpha,\ \rho_n := \rho - \rho_c.$$

Then the $\pi_\Lambda$, looked upon as a $K$-module, contains $\tau_\lambda$ with multiplicity one, and the highest weight of any $K$-type of $\pi_\Lambda$ is of the form: $\lambda + \sum_{\alpha\in\Delta^+} n_\alpha \alpha$ with integers $n_\alpha \geq 0$. We call $\tau_\lambda$ the *lowest $K$-type* of $\pi_\Lambda$.

**4. $(S(\mathfrak{g}), K)$-modules $\mathrm{Gr}\,\mathscr{A}(\tau)$.** For a finite-dimensional $K$-module $(\tau, V)$, let $\mathscr{A}(\tau)$ be the space of real analytic functions $f : G \to V$ satisfying $f(gk) = \tau(k)^{-1}f(g)$ $(g \in G, k \in K)$. The group $G$ acts on $\mathscr{A}(\tau)$ by left translation, and $\mathscr{A}(\tau)$ becomes a $U(\mathfrak{g})$-module through differentiation. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition of $\mathfrak{g}$. Setting for each integer $k \geq 0$,

(4.1) $$\mathscr{A}_{(k)} := \{f \in \mathscr{A}(\tau) \mid (X^m f)(1) = 0 \ (\forall X \in \mathfrak{p},\ 0 \leq \forall m \leq k)\}$$

and $\mathscr{A}_{(k)} := \mathscr{A}(\tau)$ for $k < 0$, one gets a decreasing $K$-stable filtration $(\mathscr{A}_{(k)})_{k\in\mathbf{Z}}$ of $\mathscr{A}(\tau)$ such that $U_m(\mathfrak{g})\mathscr{A}_{(k)} \subset \mathscr{A}_{(k-m)}$ for $k, m \geq 0$, and correspondingly we have a graded $(S(\mathfrak{g}), K)$-module

(4.2) $$\mathrm{Gr}\,\mathscr{A}(\tau) := \oplus_k \mathscr{A}_{(k)}/\mathscr{A}_{(k+1)}.$$

Now take two bases $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ to the vector space $\mathfrak{p}$ such that $B(X_i, X_j^*) = \delta_j^i$ (Kronecker's $\delta$) for the Killing form $B$ of $\mathfrak{g}$. We put

$$\iota_k(f) := \sum_{|\nu|=k+1}(1/\nu!)\cdot(X^*)^\nu \otimes (X^\nu f)(1) \in S^{k+1}(\mathfrak{p}) \otimes V \ (f \in \mathscr{A}_{(k)}),$$

where $X^{\nu} := X_1^{\nu_1} \cdots X_s^{\nu_s}$, $(X^*)^{\nu} := (X_1^*)^{\nu_1} \cdots (X_s^*)^{\nu_s}$ and $\nu! = \nu_1! \cdots \nu_s!$ for multi-indices $\nu = (\nu_1, \ldots, \nu_s)$ of length $|\nu| := \nu_1 + \cdots + \nu_s = k+1$. Observe that $\iota_k(f)$ is independent of the choice of $(X_i)_i$ and $(X_i^*)_i$, and that $\iota_k$ naturally gives a $K$-isomorphism:

(4.3) $$\tilde{\iota}_k : \mathscr{A}_{(k)} / \mathscr{A}_{(k+1)} \simeq S^{k+1}(\mathfrak{p}) \otimes V,$$

where $K$ acts on $S^{k+1}(\mathfrak{p})$ through the adjoint action.

**Lemma 1.** *The map $\tilde{\iota} := \bigoplus_k \tilde{\iota}_k$ gives a graded $(S(\mathfrak{g}), K)$-isomorphism from* $\mathrm{Gr}\,\mathscr{A}(\tau)$ *onto $S(\mathfrak{p}) \otimes V$, where $S(\mathfrak{g})$ acts on $S(\mathfrak{p}) \otimes V$ by differentiation*: $Y \cdot (X^k \otimes v) = kB(X, Y)X^{k-1} \otimes v$ *for* $Y \in \mathfrak{g}$, $X^k \otimes v \in S^k(\mathfrak{p}) \otimes V$ $(k = 0, 1, \ldots)$.

We identify $\mathrm{Gr}\,\mathscr{A}(\tau)$ with $S(\mathfrak{p}) \otimes V$ by this isomorphism $\tilde{\iota}$.

**5. Operators $\mathscr{D}_\lambda$ and graded modules Gr $H_\Lambda$.** Since the discrete series $H_\Lambda$ contains the lowest $K$-type $(\tau_\lambda, V_\lambda)$, $\lambda = \Lambda - \rho_c + \rho_n$, with multiplicity one, there exists a unique, up to scalar multiples, $(\mathfrak{g}, K)$-module embedding $H_\lambda \hookrightarrow \mathscr{A}(\tau_\lambda)$. We regard $H_\Lambda$ as a submodule of $\mathscr{A}(\tau_\lambda)$ through this embedding. Then one gets a graded $(S(\mathfrak{g}), K)$-submodule of $\mathrm{Gr}\,\mathscr{A}(\tau_\lambda)$:

$$\mathrm{Gr}\,H_\Lambda := \bigoplus_k (H_\Lambda \cap \mathscr{A}_{(k)})/(H_\Lambda \cap \mathscr{A}_{(k+1)})$$

through the decreasing filtration $\mathscr{A}_{(k)}$ of $\mathscr{A}(\tau_\lambda)$ in (4.1).

Using the bases $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ of $\mathfrak{p}$ in §4, we set for $f \in \mathscr{A}(\tau_\lambda)$,

(5.1) $$\nabla_\lambda f(g) := \textstyle\sum_{i=1}^s R_{X_i} f(g) \otimes X_i^* \quad (g \in G),$$

where $R_D$ denotes the left $G$-invariant differential operator on $G$ corresponding to $D \in U(\mathfrak{g})$. Then $\nabla_\lambda$ does not depend on the choice of dual bases, and it defines a first order, left $G$-invariant differential operator from $\mathscr{A}(\tau_\lambda)$ to $\mathscr{A}(\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}})$. Here $\mathrm{Ad}_{\mathfrak{p}}$ denotes the adjoint representation of $K$ on $\mathfrak{p}$.

Let $(\tau_\lambda^{\pm}, V_\lambda^{\pm})$ be respectively the $K$-submodules of $V_\lambda \otimes \mathfrak{p}$ generated by highest weight vectors of weights $\lambda \pm \beta$ for some $\beta \in \Delta_n^+ = \Delta_n \cap \Delta^+$, and $P_\lambda : V_\lambda \to V_\lambda^-$ be the projection along the decomposition $V_\lambda \otimes \mathfrak{p} = V_\lambda^+ \oplus V_\lambda^-$.

The above $\nabla_\lambda$, composed with $P_\lambda$ yields a $G$-invariant differential operator $\mathscr{D}_\lambda$ from $\mathscr{A}(\tau_\lambda)$ to $\mathscr{A}(\tau_\lambda^-)$:

(5.2) $$\mathscr{D}_\lambda f(g) := P_\lambda(\nabla_\lambda f(g)) \quad (f \in \mathscr{A}(\tau_\lambda)).$$

Passing to the gradation, we get an $(S(\mathfrak{g}), K)$-module map

(5.3) $$\mathrm{Gr}[\mathscr{D}_\lambda] : S(\mathfrak{p}) \otimes V_\lambda = \mathrm{Gr}\,\mathscr{A}(\tau_\lambda) \to \mathrm{Gr}\,\mathscr{A}(\tau_\lambda^-) = S(\mathfrak{p}) \otimes V_\lambda^-.$$

It follows from results of Schmid, Hotta-Parthasarathy and Wallach that the $L^2$-kernel of $\mathscr{D}_\lambda$ realizes the discrete series $\pi_\Lambda$ for each $\Lambda \in \Xi$. In order to prove Theorem 1, we employ $\mathrm{Gr}[\mathscr{D}_\lambda]$ rather than $\mathscr{D}_\lambda$ itself, and use the following

**Theorem HP** (cf. [3]). *One has* $\mathrm{Gr}\,H_\Lambda = \mathrm{Ker}(\mathrm{Gr}[\mathscr{D}_\lambda])$ *provided the lowest hightest weight $\lambda = \Lambda - \rho_c + \rho_n$ of $H_\Lambda$ is sufficiently $\Delta_c$-regular.*

**6. Outline of proof of Theorem 1.** FIRST STEP. Let $H_\Lambda^*$ be the $K$-finite dual of discrete series $H_\Lambda$. Note that $H_\Lambda^* \simeq H_{-w_0\Lambda}$ as $(\mathfrak{g}, K)$-modules, where $w_0$ is the element of Weyl group of $\Delta_c$ such that $w_0\Delta_c^+ = -\Delta_c^+$. We are going to prove

(6.1) $$\mathscr{V}(H_\Lambda^*) = K_{\mathbf{C}}\mathfrak{p}_+ = \mathrm{Ad}(K)\mathfrak{p}_+ \text{ with } \mathfrak{p}_+ := \textstyle\sum_{\beta \in \Delta_n^+} \mathfrak{g}_\beta,$$

which is equivalent to the claim of Theorem 1.

First, Theorem HP allows us to deduce the following

**Proposition 1.** *For sufficiently $\Delta_c$-regular $\lambda = \Lambda - \rho_c + \rho_n$, the associated variety $\mathcal{V}(H_\Lambda^*)$ of $H_\Lambda^*$ is expressed by means of* $\mathrm{Gr}[\mathcal{D}_\lambda]$ *as*

$$\mathcal{V}(H_\Lambda^*) = \{X \in \mathfrak{g} \mid f(X) = 0 \ (\forall f \in \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]))\}.$$

SECOND STEP. Let $v_\lambda$ be a nonzero highest weight vector of $V_\lambda$. For each integer $k \geq 0$, let $Q_k^+(\lambda)$ denote the $K$-submodule of $S^k(\mathfrak{p}) \otimes V_\lambda$ generated by subspace $S^k(\mathfrak{p}_+) \otimes v_\lambda$. Then one easily observes that

$$\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \cap (S^k(\mathfrak{p}) \otimes V_\lambda) \supset Q_k^+(\lambda). \tag{6.2}$$

We can prove the following proposition with the aid of [3, Lemma 5.2].

**Proposition 2.** *For each $k \geq 0$, there exists a constant $c_k > 0$ for which the equality holds in* (6.2) *if $(\lambda, \alpha) > c_k (\forall \alpha \in \Delta_c^+)$.*

THIRD STEP. Let $\mathcal{L}(K_C\mathfrak{p}_+) = \{f \in S(\mathfrak{g}) \mid f(X) = 0 \ (\forall \ X \in K_C\mathfrak{p}_+)\}$ be the ideal of $S(\mathfrak{g})$ defined by the cone $K_C\mathfrak{p}_+$. Noting that this ideal is finitely generated since $S(\mathfrak{g})$ is Noetherian, we deduce from Proposition 2,

**Theorem 2.** *One has $\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \subset \mathcal{L}(K_C\mathfrak{p}_+)$ for every $\lambda = \Lambda - \rho_c + \rho_n$. Moreover the equality holds in this inclusion if the parameter $\lambda$ is sufficiently $\Delta_c$-regular.*

FINAL STEP. Let $B$ be the Borel subgroup of $K_C$ with Lie algebra $\mathfrak{k} + \sum_{\alpha \in \Delta_c^+} \mathfrak{g}_\alpha$. Notice that $\mathfrak{p}_+$ is $B$-stable and that $K_C = \mathrm{Ad}(K)B$ by the Iwasawa decomposition of $K_C$. We then find that $K_C\mathfrak{p}_+ = \mathrm{Ad}(K)\mathfrak{p}_+$ is a closed subset of $\mathfrak{g}$ because of the compactness of $K$.

Now Proposition 1 and Theorem 2 yield the desired (6.1) for sufficently $\Delta_c$-regular $\lambda$. With the Zuckerman translation principle in mind (cf. [7, I, 3.4]), we conclude that (6.1) holds for every $\lambda$. This completes the proof of Theorem 1.

**7.** The above discussion leads us also to the following conclusions.

**Theorem 3.** *Assume that $\lambda$ be sufficiently $\Delta_c$-regular. Then,*

( i ) *the annihilator ideal of $S(\mathfrak{g})$-module $\mathrm{Gr}\, H_\Lambda$ coincides with its radical.*

(ii) *One has $\mathcal{V}(H_\Lambda^*) = \{X \in \mathfrak{p} \mid P_\lambda(v \otimes X) \neq 0 \ (\exists\, v \in V_\lambda \setminus (0))\}$.*

We remark that $V_\lambda \otimes \mathfrak{p} \ni (v, X) \mapsto P_\lambda(v \otimes X) \in V_\lambda^-$ is just the (complexified) symbol mapping of $\mathcal{D}_\lambda$ at the origin $o = K \in G/K$.

**8. Gelfand-Kirillov dimensions $d(H_\Lambda)$ for $SU(p, q)$.** By applying Theorem 1, we can give an explicit formula for the Gelfand-Kirillov dimensions $d(H_\Lambda) = \dim K_C\mathfrak{p}_-$ of discrete series for $G = SU(p, q)\,(n = p + q, q > 0)$.

**8.1.** Realize the group $G$ as

$$G = \{g \in SL(n, \boldsymbol{C}) \mid {}^t\bar{g}I_{p,q}g = I_{p,q}\} \text{ with } I_{p,q} = \begin{pmatrix} I_p & 0 \\ O & -I_q \end{pmatrix},$$

where $I_r$ is the identitiy matrix of degree $r$, and ${}^tg$ (resp. $\bar{g}$) denotes the transposed (resp. the complex conjugate) of a matrix $g$. Then we have $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$ and $\mathfrak{t} = \{Z = \mathrm{diag}(t_1, \ldots, t_n) \mid t_i \in \boldsymbol{C},\ \mathrm{tr}\, Z = 0\}$. The root system $\Delta$ (resp. $\Delta_c \subset \Delta$) of $\mathfrak{g}$ (resp. $\mathfrak{k}$) with respect to $\mathfrak{t}$ is of type $A_{n-1}$ (resp. $A_{p-1} \times A_{q-1}$), and it is given respectively by

$$\Delta = \{e_{ij} \mid 1 \leq i, j \leq n,\ i \neq j\},\ \Delta_c = \{e_{ij} \in \Delta \mid 1 \leq i, j \leq p \text{ or } p < i, j \leq n\}$$

with $e_{ij}(Z) := t_i - t_j (Z \in \mathfrak{t})$.

Fix a positive system $\Delta_c^+ := \{e_{ij} \in \Delta_c \mid i < j\}$ of $\Delta_c$. Let $\Pi_{p,q}$ be the totality of maps $h$ from $F(n) := \{1,2,\dots,n\}$ to the set $\{a, b\}$ of two elements $a$ and $b$, such that $\#(h^{-1}(\{a\})) = p$ and $\#(h^{-1}(\{b\})) = q$, where $\#(S)$ denotes the cardinal number of a set $S$. For an $h \in \Pi_{p,q}$, arrange the elements of $h^{-1}(\{a\})$ and $h^{-1}(\{b\})$ respectively as

$$(w(1), w(2),\dots,w(p)) \text{ with } w(1) < w(2) < \dots < w(p),$$

$$(w(p+1), w(p+2),\dots,w(n)) \text{ with } w(p+1) < w(p+2) < \dots < w(n),$$

and we put

(8.1) $$\Delta^+(h) := \{e_{ij} \in \Delta \mid w(i) < w(j)\}$$

through this $w$. Then we easily find that $h \mapsto \Delta^+(h)$ gives a one-one correspondence from $\Pi_{p,q}$ onto the set of positive systems of $\Delta$ including $\Delta_c^+$.

Now let $h \in \Pi_{p,q}$. Take a discrete series $(\mathfrak{g}, K)$-module $H_\Lambda$ with $\Delta^+(h)$-dominant parameter $\Lambda \in \Xi$. By Theorem 1, we see that $d[h] := d(H_\Lambda)$ is independent of the choice of such a $\Lambda$. The map $\Pi_{p,q} \ni h \to d[h]$ completely describes the Gelfand-Kirillov dimensions for discrete series of $G = SU(p, q)$.

We put $\Pi := \cup_{n=1}^{\infty} \Pi(n)$ (disjoint union), where the set $\Pi(n) := \cup_{p+q=n} \Pi_{p,q}$ consists of all mappings from $F(n)$ to $\{a, b\}$. Extend $h \to d[h]$, defined on each $\Pi_{p,q}$, to a function $d[\cdot]$ on $\Pi$ in the canonical way.

**8.2.** Let $h \in \Pi(n)$ $(n > 0)$. In order to specify the Gelfand-Kirillov dimension $d[h]$, we introduce an equivalence relation $\overset{h}{\sim}$ on the set $F(n)$ by

$$i \overset{h}{\sim} j \Leftrightarrow h \text{ takes the same value on the segment } [i, j].$$

Take a complete system $I_h \subset F(n)$ of representatives of the coset space $F(n)/\overset{h}{\sim}$, and let $\zeta_h : F(n) \setminus I_h \mapsto F(n - |h|)$, be the unique bijection such that

$$i < j \Leftrightarrow \zeta_h(i) < \zeta_h(j) \text{ for } i, j \in F(n) \setminus I_h,$$

where $|h| := \#(I_h)$. We define $Rh \in \Pi(n - |h|)$ by $Rh := h \circ \zeta_h^{-1}$. Note that $Rh$ is independent of the choice of a set of representatives $I_h$.

Applying $R$ repeatedly, we obtain from each $h \in \Pi(n)$ a finite sequence $(R^k(h))_{0 \le k \le l}$ of elements of $\Pi$ with

(8.2) $$R^k(h) \in \Pi(n_k(h)), \quad n_k(h) = n - \textstyle\sum_{j=0}^{k-1} |R^j(h)|.$$

Here $l$ is the non-negative integer such that $|R^l(h)| = n_l(h) > 0$.

**Theorem 4.** *The Gelfand-Kirillov dimension of an* $h \in \Pi(n) = \cup_{p+q=n} \Pi_{p,q} (n > 0)$ *is given as*

(8.3)
$$d[h] = (1/2) \cdot \sum_{k=0}^{l} (2n_k(h) - r_k)(r_k - 1) = (1/2) \cdot (n^2 - \sum_{k=0}^{l} (2k+1) r_k)$$

*with* $r_k = |R^k(h)|$, *by means of the finite sequences* $(R^k(h))_{0 \le k \le l}$ *and* $(n_k(h))_{0 \le k \le l}$ *in* (8.2).

**Example.** CASE OF $G = SU(p, 2)$ $(p \ge 2)$. In this case, the set $\Pi_{p,2}$ is divided into 7 subfamilies according to the positions of two elements $i_1, i_2 \in F(n)$ such that $h(i_1) = h(i_2) = b$, and the corresponding quantities $(r_k)_{0 \le k \le l}$ and $d[h]$ are given explicitly as follows.

| type | $(h(i))_i$ | $(r_k)_k$ | $d[h]$ |
|---|---|---|---|
| I | $(bba\ldots a)$ | $(2,2,1,\ldots,1)$ | $2p$ |
| II | $(ba\ldots aba\ldots a)$ | $(4,1,\ldots,1)$ | $3p$ |
| III | $(ba\ldots ab)$ | $(3,1,\ldots,1)$ | $2p+1$ |
| IV | $(a\ldots abba\ldots a)$ | $(3,3,1,\ldots,1)$ $(p \geq 4)$ | $4p-4$ |
| | | $(3,2)$ $(p=3)$ | 8 |
| | | $(3,1)$ $(p=2)$ | 5 |
| V | $(a\ldots aba\ldots aba\ldots a)$ $(p \geq 3)$ | $(5,1,\ldots,1)$ | $4p-2$ |
| VI | $(a\ldots aba\ldots ab)$ | $(4,1,\ldots,1)$ | $3p$ |
| VII | $(a\ldots abb)$ | $(2,2,\ldots,1)$ | $2p$ |

The details of this note will appear elsewhere.

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