66. *A Hilbert Space of Harmonic Functions*

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Introduction. Let $n = 2,3,4,\cdots$. We denote by $\mathcal{P}^k_\Delta(\boldsymbol{R}^{n+1})$ the space of k-homogeneous harmonic polynomials on $\boldsymbol{R}^{n+1}$ and by $N(k, n)$ the dimension of $\mathcal{P}^k_\Delta(\boldsymbol{R}^{n+1})$.

Ii [1] and Wada [2] introduced the following function;

$$\rho_n(r) = \begin{cases} \sum_{l=0}^{(n-1)/2} a_{nl} r^{l+1} K_l(r), & \text{if } n \text{ is odd}, \\ \sum_{l=0}^{n/2} a_{nl} r^{l+\frac{1}{2}} K_{l-\frac{1}{2}}(r), & \text{if } n \text{ is even}, \end{cases}$$

where $K_\mu(r)$, $\mu \in \boldsymbol{R}$ is the modified Bessel function and the constants a_{nl}, $l = 0,1,2,\cdots,[n/2]$ are defined uniquely by

$$\int_0^\infty r^{2k+n-1}\rho_n(r)dr = \frac{N(k, n)k!\Gamma\left(k + \frac{n+1}{2}\right)2^{2k}}{\Gamma\left(\frac{n+1}{2}\right)} \equiv C(k, n),\ k = 0,1,2,\cdots$$

(see Lemma 2.2 in [2]). Then, they constructed a Plancherel measure on the complex light cone $\tilde{M} = \{z \in \boldsymbol{C}^{n+1};\, z^2 \equiv z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0\}$, and a Hilbert space of holomorphic functions on $\tilde{M}$ by using the measure. Furthermore, they proved that the Hilbert space is unitarily isomorphic to $L^2(S^n)$ under the Fourier transformation, where S^n is the n-dimensional real sphere.

In this paper, we will construct a Hilbert space of harmonic functions on $\boldsymbol{R}^{n+1}$ and prove that the Hilbert space is unitarily isomorphic to a subspace of $L^2(M)$ under the Fourier transformation, where M is the spherical sphere:

$$M = \{z = x + iy \in \tilde{M};\, \|x\| = 1/2\} \cong O(n+1)/O(n-1).$$

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1. A Hilbert space of harmonic functions. Let $\|x\|$ be the Euclidean norm on $\boldsymbol{R}^{n+1}$. We denote by $\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$ the space of harmonic functions on $\boldsymbol{R}^{n+1}$ equipped with the topology of uniform convergence on compact sets. Define the k-homogeneous harmonic component f_k of $f \in \mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$ by

$$(1)\qquad f_k(z) = N(k, n)(\sqrt{z^2})^k \int_{S^n} f(\tau) P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\tau\right) dS(\tau), \quad z \in \boldsymbol{C}^{n+1},$$

where $z\cdot w = \sum_{j=1}^{n+1} z_j w_j$, z, $w \in \boldsymbol{C}^{n+1}$, $P_{k,n}(t)$ is the Legendre polynomial of degree k and of dimension $n+1$, and dS is the normalized $O(n+1)$-invariant measure on S^n.

The following lemmas are known:

Lemma 1. *Let $f \in \mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$ and f_k be the k-homogeneous harmonic component of f defined by* (1). *Then the expansion $\sum_{k=0}^\infty f_k$ converges to f in the topology of $\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$.*

Lemma 2. *Let* $f_k \in \mathcal{P}_\Delta^k(\boldsymbol{R}^{n+1})$ *and* $f_l \in \mathcal{P}_\Delta^l(\boldsymbol{R}^{n+1})$. *If* $k \neq l$, *then*

$$\int_{S^n} f_k(\omega) g_l(\omega) dS(\omega) = 0.$$

We define a measure $d\mu$ on $\boldsymbol{R}^{n+1}$ by

$$\int_{\boldsymbol{R}^{n+1}} f(x) d\mu(x) = \int_0^\infty \int_{S^n} f(r\omega) dS(\omega) r^{n-1} \rho_n(r) dr.$$

Note that $\rho_n(r)$ is not positive but there is $R_n > 0$ such that $\rho_n(r) > 0$ for $r > R_n$. The function ρ_n is estimated as follows:

$$\text{(2)} \qquad \begin{cases} |\rho_n(r)| \leq e^{-r} r^{1/2} P_{(n-1)/2}(r), & \text{if } n \text{ is odd,} \\ |\rho_n(r)| = e^{-r} P_{n/2}(r), & \text{if } n \text{ is even,} \end{cases}$$

where $P_{n/2}(r)$ and $P_{(n-1)/2}(r)$ are polynomials of degree $[n/2]$ (see [1], p. 64 and [2], p. 429).

We define a sesquilinear form $(f, g)_{\boldsymbol{R}^{n+1}}$ by

$$(f, g)_{\boldsymbol{R}^{n+1}} = \int_{\boldsymbol{R}^{n+1}} f(x) \overline{g(x)} d\mu(x).$$

Although $\rho_n(r)$ is not positive, the sesquilinear form $(f, g)_{\boldsymbol{R}^{n+1}}$ is an inner product on

$$L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1}) = \{f \in \mathcal{A}_\Delta(\boldsymbol{R}^{n+1})\,;\, \|f\|^2_{\boldsymbol{R}^{n+1}} \equiv (f, f)_{\boldsymbol{R}^{n+1}} < \infty\}$$

by the following proposition:

Proposition 3. *Let* $f = \sum f_k \in \mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$\begin{aligned}(f, f)_{\boldsymbol{R}^{n+1}} &= \sum_{k=0}^\infty (f_k, f_k)_{\boldsymbol{R}^{n+1}} \\ &= \sum_{k=0}^\infty C(k, n) \int_{S^n} f_k(\omega) \overline{f_k(\omega)} dS(\omega) \geq 0,\end{aligned}$$

i.e., either both sides are infinite or both sides are finite and equal.

Proof. For $R > 0$ we put $C_R(k, n) = \int_0^R r^{2k+n-1} \rho_n(r) dr$ and

$$I(R) = \int_0^R \int_{S^n} f(r\omega) \overline{f(r\omega)} dS(\omega) r^{n-1} \rho_n(r) dr.$$

Since $\rho_n(r) > 0$ for $r > R_n$, $I(R)$ is monotone increasing for $R > R_n$ and $(f, f)_{\boldsymbol{R}^{n+1}} = \lim_{R \to \infty} I(R)$. By Lemma 1,

$$I(R) = \sum_{k=0}^\infty \frac{C_R(k, n)}{C(k, n)} (f_k, f_k)_{\boldsymbol{R}^{n+1}}.$$

Similar to the proof of Proposition 2.4 in [1], we can prove

$$\lim_{R \to \infty} \sum_{k=0}^\infty \frac{C_R(k, n)}{C(k, n)} (f_k, f_k)_{\boldsymbol{R}^{n+1}} = \sum_{k=0}^\infty (f_k, f_k)_{\boldsymbol{R}^{n+1}}.$$

Q.E.D.

Put

$$\text{(3)} \quad E^s(\boldsymbol{R}^{n+1}) = \{f \in \mathcal{A}_\Delta(\boldsymbol{R}^{n+1})\,;\, \exists\, C > 0, |f(x)| \leq C e^{s\|x\|}, \ \forall x \in \boldsymbol{R}^{n+1}\}.$$

By (2), it is easy to see that for s with $0 \leq s < 1/2$

$$E^s(\boldsymbol{R}^{n+1}) \subset L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1}) \subset E^{1/2}(\boldsymbol{R}^{n+1}).$$

For $f \in \mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$ and $0 \leq t \leq 1$, put $f^t(x) = f(tx)$.

Lemma 4. (i) $f \in L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$ *if and only if* $f^t \in L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$, $0 < t < 1$ *and* $\sup\{\|f^t\|_{\boldsymbol{R}^{n+1}}\,;\, 0 < t < 1\} < \infty$.

(ii) *Let* $f \in L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then* $\lim_{t \uparrow 1} \|f - f^t\|_{\boldsymbol{R}^{n+1}} = 0$.

Proof is similar to that of Lemma 2.8 in [1] and is omitted.

Put

$$
\begin{aligned}
E_1(x, y) &= \int_M \exp(\zeta\cdot x)\exp(y\cdot\bar{\zeta})dM(\zeta) \\
&= \sum_{k=0}^{\infty} \frac{N(k, n)}{C(k, n)} \|x\|^k \|y\|^k P_{k,n}\Big(\frac{x}{\|x\|}\cdot\frac{y}{\|y\|}\Big).
\end{aligned} \tag{4}
$$

Then $E_1(x, y)$ is real valued, symmetric and satisfies

$$
\Delta_x E_1(x, y) \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n+1}^2}\Big)E_1(x, y) = 0. \tag{5}
$$

By (4), we have the following estimation: there is a constant C such that $|E_1(x, y)| \leq Ce^{\|x\|/(2A)}e^{A\|y\|/2}$ for any $A > 0$. In particular, $E_1(x, \cdot)$ and $E_1(\cdot, y)$ belong to $\in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$.

Theorem 5. *Let* $f \in L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$
f(y) = (f_x, E_1(y, x))_{\boldsymbol{R}^{n+1}} = \int_{\boldsymbol{R}^{n+1}} f(x)\overline{E_1(y, x)}d\mu(x),\ y \in \boldsymbol{R}^{n+1}.
$$

By Proposition 3, (4) and (5), an easy computation completes the proof.

We denote by X_k the space $\mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$ with the inner product given by

$$
(f_k, g_k) = C(k, n)\int_{S^n} f_k(\omega)\overline{g_k(\omega)}dS(\omega).
$$

Put $$E_{1,k}(x, y) = \frac{N(k, n)}{C(k, n)} \|x\|^k \|y\|^k P_{k,n}\Big(\frac{x}{\|x\|}\cdot\frac{y}{\|y\|}\Big).$$

Then $E_{1,k}(\cdot, y) \in \mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1}) \subset L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1})$, and $f_k(y) = (f_k(x), E_{1,k}(y, x))_{\boldsymbol{R}^{n+1}}$ by Theorem 5, (4) and Lemma 2. Thus X_k is an $N(k, n)$-dimensional Hilbert space with the reproducing kernel $E_{1,k}$. We can prove that the direct sum decomposition of the Hilbert space $L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}) : L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}) = \oplus_{k=0}^{\infty} X_k$.

Thus we have the following theorem:

Theorem 6. $(L^2\mathscr{A}_\Delta(\boldsymbol{R}^{n+1}), (\ ,\ )_{\boldsymbol{R}^{n+1}})$ *is a Hilbert space with the reproducing kernel* E_1.

2. The Fourier transformation. We denote by $L^2(M)$ the space of square integrable functions on M with the inner product given by

$$
(f, g)_M = \int_M f(w)\overline{g(w)}dM(w),
$$

where dM is the normalized $\mathrm{O}(n+1)$-invariant measure on M.

We define the k-homogeneous component F_k of $F \in L^2(M)$ by

$$
F_k(z) = 2^k N(k, n)\int_M F(w)(z\cdot\bar{w})^k dM(w),\ z \in \mathrm{C}^{n+1}. \tag{6}
$$

Then $F_k|_{\boldsymbol{R}^{n+1}} \in \mathscr{P}^k_\Delta(\boldsymbol{R}^{n+1})$. We denote by $\mathscr{P}^k(\tilde{M})$ the space of the k-homogeneous polynomials on $\tilde{M}$. For $F_k \in \mathscr{P}^k(\tilde{M})$, it is known that

$$
F_k(z) = \delta_{kl}2^l N(k, n)\int_M F_k(w)(z\cdot\bar{w})^l dM(w),\ z \in \tilde{M}
$$

(Lemma 1.3 in [2]). We denote by $\mathscr{O}(\tilde{M}[1])$ the space of germs of holomorphic functions on $\tilde{M}[1] = \{z = x + iy \in \tilde{M}\,;\, \|x\| \leq 1/2\}$. Let $L^2\mathscr{O}(M)$ be the closure of $\mathscr{O}(\tilde{M}[1])|_M$ in $L^2(M)$. Then $L^2\mathscr{O}(M)$ is a closed subspace of $L^2(M)$ and the following lemma is clear:

Lemma 7. *Let* $F \in L^2\mathscr{O}(M)$ *and* F_k *be the* k*-homogeneous component of* F

defined by (6). *Then the expansion* $\sum_{k=0}^{\infty} F_k$ *converges to* F *in the topology of* $L^2\mathcal{O}(M)$.

Lemma 8. (cf. [1, Lemma 1.7] or [2, Lemma 1.4]). *Let* $f_k \in \mathcal{P}_\Delta^k(\boldsymbol{R}^{n+1})$ *and* $f_l \in \mathcal{P}_\Delta^l(\boldsymbol{R}^{n+1})$. *Then*

$$\delta_{kl} \int_{S^n} f_k(\omega)\overline{g_l(\omega)}\,dS(\omega) = \frac{N(k,\, n)\Gamma\left(\frac{n+1}{2}\right)k!}{\Gamma\left(k+\frac{n+1}{2}\right)} \int_M f_k(w)\overline{g_l(w)}\,dM(w).$$

We define the Fourier transform $\mathcal{F}F$ of $F \in L^2(M)$ by

$$\mathcal{F}F(x) = \int_M F(w)\exp(x\cdot\bar{w})\,dM(w),\ x \in \boldsymbol{R}^{n+1}.$$

Then

$$\mathcal{F}F(x) = \sum_{k=0}^{\infty} \frac{1}{N(k,\, n)k!2^k} F_k(x),\ x \in \boldsymbol{R}^{n+1}. \tag{7}$$

Theorem 9. $\mathcal{F} : F \mapsto \mathcal{F}F$ *is a unitary isomorphism of* $L^2\mathcal{O}(M)$ *onto* $L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$.

Proof. Let $F \in L^2\mathcal{O}(M)$. By Lemmas 7, 8 and (7),

$$\begin{aligned}\infty > (F,\, F)_M &= \sum_{k=0}^{\infty} \int_M F_k(w)\overline{F_k(w)}\,dM(w)\\ &= \sum_{k=0}^{\infty} C(k,\, n) \int_{S^n} \frac{F_k(\omega)}{N(k,\, n)k!2^k}\,\overline{\frac{F_k(\omega)}{N(k,\, n)k!2^k}}\,dS(\omega)\\ &= (\mathcal{F}F,\, \mathcal{F}F)_{\boldsymbol{R}^{n+1}}.\end{aligned}$$

Thus $\mathcal{F}$ is an isometric mapping of $L^2\mathcal{O}(M)$ into $L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$.

Surjectivity of F can be proven by Proposition 3, Lemmas 2 and 8.

Q.E.D.

Theorem 10. *If* $f \in E^s(\boldsymbol{R}^{n+1})$, $0 \leq s < 1/2$, *then*

$$\mathcal{F}^{-1}f(z) = \int_{\boldsymbol{R}^{n+1}} \exp(x\cdot z)f(x)\,d\mu(x),\ z \in M. \tag{8}$$

Proof. The right-hand side in (8) converges absolutely by (2) and (3), which we denote by $F(z)$. Then by the Fubini theorem and Theorem 5, $\mathcal{F}F(x) = f(x)$. **Q.E.D.**

Corollary 11. *Let* $f \in L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$\mathcal{F}^{-1}f(z) = \underset{t\uparrow 1}{\text{l.i.m.}} \int_{\boldsymbol{R}^{n+1}} \exp(x\cdot z)f(tx)\,d\mu(x),\ z \in M,$$

where l.i.m. *means the strong convergence in* $L^2(M)$.

Theorem 12. *Let* $f \in L^2\mathcal{A}_\Delta(\boldsymbol{R}^{n+1})$. *Then*

$$\mathcal{F}^{-1}f(z) = \underset{R\to\infty}{\text{l.i.m.}} \int_0^R \left(\int_{S^n} \exp(r\omega\cdot z)f(r\omega)\,dS(\omega)\right)r^{n-1}\rho_n(r)\,dr,\ z \in M.$$

Proof is similar to that of Theorem 2.11 in [1] and is omitted.

References

[1] K. Ii: On a Bargmann-type transform and a Hilbert space of holomorphic functions. Tôhoku Math. J., **38**, 57–69 (1986).

[2] R. Wada: On the Fourier-Borel transformations of analytic functionals on the complex sphere. ibid., **38**, 417–432 (1986).