6. On Some Foliations on Ruled Surfaces

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§0. Introduction. Every ruled surface has a foliation – the ruling –, which characterizes ruled surfaces in all compact complex surfaces. Existence of another foliation characterizes some ruled surfaces in all ruled surfaces. In this paper, we classify ruled surfaces with a foliation on them leaving a curve invariant and having no singularities on it. §1 is a short review of ruled surfaces. The main theorem is stated in §2. To prove it, we need the index formula of Camacho-Sad, which we review in §3. The examples are given in §4. The details of the proof etc. will be found in [10]. The author thanks Prof. T. Suwa for his helpful advices.

§1. A review of ruled surfaces. In this section, we review some properties of ruled surfaces, which may be found in eg. [6].

Definition 1.0. *A ruled surface* $X \xrightarrow{\pi} C$ is a proper holomorphic map of a two-dimensional compact complex manifold X onto a closed Riemann surface C which makes X a $\boldsymbol{P}^1$-bundle over C.

Proposition 1.1. 0) *A ruled surface has a section, i. e. there exists a holomorphic map* $C \xrightarrow{\sigma} X$ *satisfying* $\pi \cdot \sigma = id_C$.

1) *For a ruled surface* $X \xrightarrow{\pi} C$, *there exists a section* C_0 *with the following properties*:

$C_0^2 =$ *the minimum of self-inersection numbers of sections of* $X \xrightarrow{\pi} C$.

We define a number e by

$$e = - C_0^2, \tag{1.2}$$

which satisfies the following inequality

$$e \geq - g, \tag{1.3}$$

where g is the genus of the Riemann surface C.

For a ruled surface $X \xrightarrow{\pi} C$, the exponential sequences

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on X and C induce the following commutative diagram of the cohomology long exact sequences.

$$\begin{array}{ccccccccc} \to & \mathrm{H}^1(X, \mathcal{O}_X) & \to & \mathrm{H}^1(X, \mathcal{O}_X^*) & \xrightarrow{c} & \mathrm{H}^2(X, \mathbf{Z}) & \to & \mathrm{H}^2(X, \mathcal{O}_X) \\ & \pi^* \uparrow & & \pi^* \uparrow & & \pi^* \uparrow & & \pi^* \uparrow \\ \to & \mathrm{H}^1(C, \mathcal{O}_C) & \to & \mathrm{H}^1(X, \mathcal{O}_C^*) & \xrightarrow{c} & \mathrm{H}^2(X, \mathbf{Z}) & \to & \mathrm{H}^2(X, \mathcal{O}_X) \end{array}$$

We adopt the following notations:

$$\mathrm{Pic}_0 C = ker[\mathrm{H}^1(C, \mathcal{O}_C^*) \xrightarrow{c} \mathrm{H}^2(C, \mathbf{Z})] \text{ and}$$
$$\mathrm{Pic}_0 X = ker[\mathrm{H}^1(X, \mathcal{O}_X^*) \xrightarrow{c} \mathrm{H}^2(X, \mathbf{Z})].$$

Since

$$\mathrm{H}^2(C, \mathcal{O}_C) = 0 \text{ and } \mathrm{H}^2(X, \mathcal{O}_X) = 0,$$

we have the following exact commutative diagram.

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \rightarrow & \mathrm{Pic}_0 X & \rightarrow & \mathrm{H}^1(X, \mathcal{O}_X^*) & \xrightarrow{c} & \mathrm{H}^2(X, \mathbf{Z}) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & \mathrm{Pic}_0 C & \rightarrow & \mathrm{H}^1(C, \mathcal{O}_C^*) & \xrightarrow{c} & \mathrm{H}^2(C, \mathbf{Z}) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{1.4}$$

We denote by Num$X$ the group of numerically equivalent classes of divisors on X. This induces the following isomorphism.

$$\begin{array}{rcccc} \mathrm{Num}X & \simeq & \mathrm{H}^2(X, \mathbf{Z}) & & \\ & = & \mathbf{Z}c(C_0) & \oplus & \pi^*\mathrm{H}^2(C, \mathbf{Z}) \\ & = & \mathbf{Z}c(f) & \oplus & \mathbf{Z}c(f) \\ & \simeq & \mathbf{Z} & \oplus & \mathbf{Z} \end{array}$$

Here $c(C_0)$ and $c(f)$ are images by the Chern map $\mathrm{H}^1(X, \mathcal{O}_X^*) \xrightarrow{c} \mathrm{H}^2(X, \mathbf{Z})$ of the holomorphic line bundles defined by the divisors C_0, the section of $X \xrightarrow{\pi} C$, and f, the fibre of $X \xrightarrow{\pi} C$, respectively. We often denote them by C_0 and f. Thus we have the following intersection relations:

$$C_0^{\ 2} = -e, \ C_0 \cdot f = 1 \text{ and } f^2 = 0. \tag{1.5}$$

Proposition 1.6 (cf. [6] p. 382). *Let $X \xrightarrow{\pi} C$ and C_0 be as above.*

I) *The case $e \geq 0$. If an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ on X is neither C_0 nor a fibre of $X \xrightarrow{\pi} C$ then*

$$a \geq 1 \text{ and } b \geq ea.$$

II) *The case $e < 0$.*

II-0) *If an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ on X is a section of $X \xrightarrow{\pi} C$ then*

$$a = 1 \text{ and } b \geq 0.$$

II-1) *If an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ on X is neither a section nor a fibre of $X \xrightarrow{\pi} C$ then*

$$a \geq 2 \text{ and } b \geq \frac{1}{2} ea.$$

Here, "$\simeq_{num}$" represents the numerically equivalence of divisors on X.

§2. The statement of the main theorem. A foliation of dimension one can be defined in various ways (cf. [3], [4], [7], [8], and [9]). In this paper, we adopt the following one.

Let M be a complex manifold of dimension m, $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M and Θ_M the sheaf of germs of holomorphic vector fields on M.

Definition 2.0. 0) *A foliation of dimension one* on M is an invertible subsheaf $\mathcal{F}$ of Θ_M with the following property. The analytic set

$$\{p \in M \mid (\Theta/\mathcal{F})_p \text{ is } \textit{not} \text{ a free } \mathcal{O}_p\text{-module of rank } m-1\},$$

which is called the *singular locus* of $\mathcal{F}$, is of codimension strictly greater thah one.

1) A submanifold N of M defined by a coherent ideal sheaf $\mathscr{I} \subset \mathcal{O}_M$ is said to be *invariant* with respect to a foliation $\mathscr{F} \subset \Theta_M$ on M if, at every $p \in M$,

$$\mathscr{F}_p \mathscr{I}_p \subset \mathscr{I}_p.$$

In what follows, we always assume that X is a ruled surface $X \xrightarrow{\pi} C$ with the invariant e, where C is a closed Riemann surface of genus g unless otherwise stated explicitly. We fix a section C_0 of $X \xrightarrow{\pi} C$ satisfying $C_0^{\ 2} = -e$. Let f be a fibre of $X \xrightarrow{\pi} C$. We have

$$\mathrm{H}^2(X, \mathbf{Z}) = \mathbf{Z}C_0 \oplus \mathbf{Z}f \simeq \mathbf{Z}^2,$$
$$C_0^2 = -e, \ C_0 \cdot f = 1 \text{ and } f^2 = 0.$$

Main Theorem 2.1. *Let $X \xrightarrow{\pi} C$, C_0 and f be as above. Assume that a foliation $\mathscr{F} \subset \Theta_X$ on X leaves an irreducible curve $C_1 \simeq_{num} aC_0 + bF$ with $a > 0$ on X invariant and has no singularities on C_1. Then one of the following is the case.*

I) $e = 0$ *and* $b = 0$.

II) $e < 0$, $a \geq 2$ *and* $b = \dfrac{1}{2} ea \in \mathbf{Z}$.

To prove this theorem, Proposition 1.6 and the index formula of Camacho-Sad [2], of which we make a brief review, are of essential importance.

§3. The index formula of Camacho-Sad and the proof of the theorem. Let M be a complex surface, i.e. a complex manifold of dimension 2, $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M and Θ_M the sheaf of germs of holomorphic vector fields on M. Assume a foliation $\mathscr{F} \subset \Theta$ leaves a compact curve N in M invariant. Take an open neighbourhood $U \subset M$ of $q \in N$ with a local coordinate (x,y) such that $N \cap U = \{y = 0\}$, $q = (0,0)$ and that $\mathscr{F}|_U = \mathcal{O}_U \theta$, where

$$\theta = u(x, y) \frac{\partial}{\partial x} + v(x, y) \frac{\partial}{\partial y} \in \Gamma(U, \Theta).$$

Definition 3.0. The *index of $\mathscr{F}$ at q with respect to N* is

$$i_q(\mathscr{F}, N) = \mathrm{Res}_{x=0} \frac{\partial}{\partial y} \left(\frac{v}{u}\right)(x, 0)\, dx.$$

If $q \in N$ is not a singular point of $\mathscr{F}$ then $i_q(\mathscr{F}, N) = 0$. Camacho-Sad's index formula is as follows: ([2] p. 592.)

Theorem 3.1 (Camacho-Sad).

$$\sum_{q \in N} i_q(\mathscr{F}, N) = N^2.$$

Proof of Theorem 2.1. All notations are as in Theorem 2.1. Since that the foliation $\mathscr{F} \subset \Theta$ leaves the curve $C_1 \simeq_{num} aC_0 + bf$ invariant and that $\mathscr{F}$ has no singularities on C_1, the index formula asserts that

$$C_1^{\ 2} = a(2b - ea) = 0.$$

Thus $2b = ea$. (Note that $a > 0$.) It follows from Proposition 1.6 that

I) if $e \geq 0$ then $b = 0$ and $e = 0$ and that

II) if $e < 0$ then $a \geq 2$ and $b = \dfrac{1}{2} ea \in \mathbf{Z}$.

§4. Examples. In this section, we give examples of the cases stated in

Theorem 2.1. It should be noted that a foliation $\mathscr{F} \subset \Theta_X$ on a complex manifold X defines, by taking local generators of $\mathscr{F}$, a morphism $L \xrightarrow{\varphi} TX$ of holomorphic vector bundles over X of a holomorphic line bundle L into the holomorphic tangent bundle TX, which also we call a foliation. The zero-locus $\{\varphi = 0\}$, which is the *singular locus* of the foliation $L \xrightarrow{\varphi} TX$, is of codimension strictly greater than one. Since the complex manifold X is, in our case, a ruled surface, every holomorphic line bundle over X is meromorphically trivial. Thus the foliation $L \xrightarrow{\varphi} TX$ defines a global meromorphic vector field on X except for multiplication of global meromorphic functions. We display examples by assigning global meromorphic vector fields. All notations are as in Theorem 2.1.

Case I. We consider the case $g \geq 1$ and $X = \boldsymbol{P}(\mathscr{E})$, where $\mathscr{E} = \mathscr{O}_C \oplus \mathscr{L}$ is a locally free $\mathscr{O}_C$-module of rank 2 with an invertible sheaf $\mathscr{L}$ satisfying $deg\mathscr{L} = -e$. Since $e = 0$, we can take a coordinate covering $\{(U_\alpha, z_\alpha)\}$ of C such that the correspondent vector bundle E is represented by a 1-cocycle $(E_{\alpha\beta})$ of the form

$$E_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & d_{\alpha\beta} \end{bmatrix}$$

with $0 \neq d_{\alpha\beta} \in \boldsymbol{C}$. As a normalized section C_0 of $X \xrightarrow{\pi} C$, either $z_\alpha = 0$ or $z_\alpha = \infty$ does. Take a constant $c \in \boldsymbol{C}$ and a global meromorphic vector field $v \in \Gamma(C, \mathscr{M}(TC))$ with no zero arbitrary. Such a vector field v always exists. For a global holomorphic 1-form $(u_\alpha dz_\alpha) \in \Gamma(C, \mathscr{O}_C(T^*C))$, $\left(\frac{1}{u_\alpha}\frac{d}{dz_\alpha}\right) \in \Gamma(C, \mathscr{M}(TC))$is the desired one. On each U_α, we define a meromorphic vector field

$$\frac{1}{u_\alpha}\frac{\partial}{\partial z_\alpha} + c\zeta_\alpha\frac{\partial}{\partial \zeta_\alpha} \in \Gamma(U_\alpha \times \boldsymbol{P}^1, \mathscr{M}(TX)).$$

These vector fields patch together to define a global meromorphic vector field $\theta \in \Gamma(X, \mathscr{M}(TX))$, which defines a foliation on X leaving C_0 invariant and having no singularities on C_0.

Case II. We consider the case that the genus g of C is one. Since $e \geq -g$, we have $e = -1$. We construct an example of the case $a = 2$ and $b = -1$. Let C^- be an elliptic curve with periods $(2\omega_1, 2\omega_2)$ and $W = C^- \times \boldsymbol{P}^1$. We denote by $\wp$ the Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_2)$ and define an elliptic function $\sigma(w)$ by

$$\sigma(w) = \frac{\wp'(w)}{2(\alpha_3 - \alpha_2)^{\frac{1}{2}}(\wp(w) - \alpha_1)},$$

where $\alpha_1 = \wp(\omega_1)$, $\alpha_2 = \wp(\omega_2)$ and $\alpha_3 = \wp(\omega_1 + \omega_2)$. $\sigma(w)$ defines a section of $W \to C^-$. Let G be a subgroup of the group of holomorphic automorphisms of W generated by

$$\begin{array}{ccc} W = C^- \times \boldsymbol{P}^1 & \longrightarrow & W = C^- \times \boldsymbol{P}^1 \\ ([w], \xi) & \mapsto & ([w + \omega_1], -\xi) \end{array}$$

and

$$W = C^- \times \boldsymbol{P}^1 \longrightarrow W = C^- \times \boldsymbol{P}^1$$
$$([w],\ \xi) \qquad \mapsto \quad ([w + \omega_2],\ \frac{1}{\xi}).$$

The quotient space $X = W/G$ is a ruled surface over an elliptic curve C with periods (ω_1, ω_2). The section of $W \to C^-$ defined by σ defines a section of $X \xrightarrow{\pi} C$, which we denote $C \xrightarrow{\sigma_0} X$. $C_0 = \sigma_0(C)$ is a normalized section. We have $C_0^2 = -1$ and $C_0 \cdot f = 1$. Let C_1 be a curve in X defined by curves in W with equations $\xi = 0$ or ∞. As a divisor on X, $C_1 \simeq_{num} 2C_0 - f$. $\frac{\partial}{\partial w} \in \Gamma(W, \Theta_W)$ defines a foliation $\mathscr{F} \subset \Theta_W$ on W, which defines a foliation on X leaving C_1 invariant and having no singularities on C_1. This is the desired one.

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