# 14. *A Generalization of Tate-Nakayama Theorem by Using Hypercohomology*

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**§1. Introduction and notations.** In the classical class field theory, the isomorphism theorem is proved by using Tate's cohomology and Tate-Nakayama theorem [1], and similar methods are used to prove the isomorphism theorem of higher dimensional class field theory (cf. [2], [3] and [4]). Especially, the proof of the isomorphism theorem of class field theory of two dimensional local fields, as given in [2], looks like the classical one by using generalized Tate-Nakayama theorem and modified hypercohomology, which is a very satisfactory generalization of Tate's cohomology.

For higher dimensional class field theory, further generalization of Tate-Nakayama theorem seems to be of great interest. This was partially achieved in [2], where this theorem was proved for two-term complexes. The aim of this paper is to prove it for arbitrary bounded complexes.

Unless the contrary is explicitly stated, we shall employ the following notation and convention throughout this paper: all groups are finite and all complexes are bounded. Let $G$ be a group and $M$ a $G$-module. We denote $M^G$ by $\Gamma(G, M)$, which is viewed as a functor. We shall freely use the standard notations on complexes and objects in derived categories as in [2], [3] and [4]. For example, for a complex $A^{\bullet}$ and an integer $m$, we define a new complex $A^{\bullet}[m]$ by $(A[m])^q = A^{q+m}$.

**§2. The generalized Tate-Nakayama theorem.** As a preparation, we recall the definition and basic properties of modified hypercohomology.

Consider an exact sequence

$$\cdots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to X^2 \to \cdots$$

such that

(1) Each term $X^n$ is a free $\mathbf{Z}[G]$-module with a finite basis.

(2) The sequence

$$\cdots \to X^{-2} \to X^{-1} \to X^0 \to \mathbf{Z} \to 0$$

is a projective resolution of the $G$-module $\mathbf{Z}$ with trivial action.

Such an exact sequence is called a *complete resolution* of $G$.

It is a well-known fact that for any $G$-module $M$, the cohomology groups of the complex:

$$\cdots \to \mathrm{Hom}_G(X^1, M) \to \mathrm{Hom}_G(X^0, M) \to \mathrm{Hom}_G(X^{-1}, M) \to \cdots$$

coincides with Tate's cohomology groups.

Note that in the definitions of usual hypercohomology, we consider the double complex $\bigoplus_{i \geq 0} Y^{i,j}$ such that

$$Y^{i,j} = \mathrm{Hom}_G(X^{-j}, A^j),$$

where the sign rule of the differentials is suitably determined. We shall denote the cohomology groups of the single complex associated to this complex $Y^{i,j}$ by $\hat{H}^*(G, A^\cdot)$. This will be called *modified hypercohomology* of $G$ with coefficient $A^\cdot$. Note that abelian groups $\hat{H}^q(G, A^\cdot)$ are uniquely determined by a complex of $G$-modules $A^\cdot$ up to isomorphisms, which are independent of a choice of complete resolutions of $G$.

**Proposition 1** (Prop. 1.2. of [2]). *Let $A^\cdot$ be a bounded complex of $G$-modules such that $A^q = 0$ for any positive integer $q$. Then we have*

(1) $$\hat{H}^0(G, A^\cdot) \simeq \mathrm{Coker}(\mathscr{H}^0(A^\cdot) \xrightarrow{N_G} H^0(G. A^\cdot)),$$

*where the homomorphism $N_G$ is the norm map of hypercohomology of groups.*

(2) *For any integer $q \geq 1$, the $q$-th modified hypercohomology groups coincides with the usual $q$-th hypercohomology of groups.*

**Theorem 2** (Thm. 1.3 of [2]). *For a triangle of complexes of $G$-module*

$$A^\cdot \to B^\cdot \to C^\cdot \to A^\cdot[1],$$

*there is a long exact sequence of modified hypercohomologies as follows:*

$$\cdots \to \hat{H}^q(G, A^\cdot) \to \hat{H}^q(G, B^\cdot) \to \hat{H}^q(G, C^\cdot) \to \hat{H}^{q+1}(G, A^\cdot) \to \cdots.$$

**Lemma 3.** *For any bounded complex of $G$-modules $A^\cdot$, one can find complex $A_\dagger{}^\cdot$ and $A^{\dagger\cdot}$ such that*

(1) $\hat{H}^{q+1}(G, A_\dagger{}^\cdot) = \hat{H}^q(G, A^\cdot)$

(2) $\hat{H}^{q-1}(G, A^{\dagger\cdot}) = \hat{H}^q(G, A^\cdot)$.

*Proof.* We have only to prove (1), as (2) can be proved similarly by taking dual. We define the complex $B^\cdot$ by $B^q = \boldsymbol{Z}[G] \otimes A^q$, where the action of $G$ on each term $B^q$ is defined as follows:

$$\tau(\sigma \otimes a) = \tau\sigma \otimes \tau a \ (\sigma, \tau \in G, a \in A^q).$$

So we have the natural morphism of complexes of $G$-module $B^\cdot \to A^\cdot$ and distinguished triangle

(3.1) $$B^\cdot \to A^\cdot \to C^\cdot \to B^\cdot[1],$$

where $C^\cdot$ is the mapping cone of the above morphism. Now we set $A_\dagger{}^\cdot = C^\cdot[1]$. Since the group $G$ is finite and $\boldsymbol{Z}[G]$ is a free $G$-module of finite rank, we have $\mathscr{H}^q(B^\cdot) = \boldsymbol{Z}[G] \otimes \mathscr{H}^q(A)$ for any integer $q$. As is well-known, $G$-modules $\boldsymbol{Z}[G] \otimes \mathscr{H}^q(A)$ are cohomologically trivial. Therefore, from the hypercohomology spectral sequence, we see that $\hat{H}^q(G, B^\cdot)$ for every integer $q$. Noting Theorem 2 and distinguished triangle (3.1), we have $\hat{H}^{q+1}(G, A_\dagger{}^\cdot) = \hat{H}^q(G, A^\cdot)$. Q. E. D.

We call a complex of $G$-modules $A^\cdot$ *cohomologically trivial* if $\hat{H}^q(H, A^\cdot) = 0$ for every integer $q$ and every subgroup $H$ of $G$. When a complex $A^\cdot$ is a usual $G$-module, this definition is compatible with the classical definition.

The next proposition is a generalization of the "twin number criterion" (cf. [6], Chap. V, §2, Theorem. 31).

**Proposition 4** (Generalized twin number criterion). *Let $G$ be a finite group and $A^\cdot$ a bounded complex of $G$-modules. Then the following are equivalent:*

(1) *For every $p$-Sylow subgroup $G_p$, there exist two consecutive integers $i_p$, $i_p + 1$ such that*

$$\hat{H}^{i_p}(G_p, A^\cdot) = \hat{H}^{i_p+1}(G_p, A^\cdot) = 0.$$

(2) $A^{\bullet}$ *is cohomologically trivial.*

*Proof.* (2)$\Rightarrow$(1) is trivial. So we have only to prove (1)$\Rightarrow$(2).

First we assume $A^q = 0$ for any positive integer $q$. We shall prove the proposition by induction on the order of $G_p$. If the order of $G_p$ is 1, the proposition is clear. Let $n$ be the order of $G_p$. And we assume that the proposition holds for all groups of order less than $n$. By an elementary property of finite $p$-groups, we can find a normal subgroup $N(\neq G_p)$ of $G_p$ such that $G_p/N$ is cyclic (cf. [5], Chap. IX, §1, Cor. to Théorème 1).

Noting that

$$H^q(G_p, A^{\bullet}) = H^q(G_p/N, R\Gamma(N, A^{\bullet}))$$

and the hypercohomology spectral sequence

$$H^p(G_p/N, \mathscr{H}^q(R\Gamma(N, A^{\bullet}))) \Rightarrow H^{p+q}(G_p/N, R\Gamma(N, A^{\bullet})),$$

we have the following isomorphism

(4.1) $$H^p(G_p, A^{\bullet}) = H^q(G_p/N, H^0(N, A^{\bullet})),$$

for $q \geq 0$. Here we use the fact that $\mathscr{H}^q(R\Gamma(N, A^{\bullet})) = H^q(N, A^{\bullet})$ and the consequence of the assumption of induction, which states that $H^q(N, A^{\bullet}) = 0$ for every integer $q$.

By the assumptions of the proposition and (4.1), we see

$$\hat{H}^{i_p}(G_p/N, H^0(N, A^{\bullet})) = \hat{H}^{i_p+1}(G_p/N, H^0(N, A^{\bullet})) = 0.$$

Since $G_p/N$ is cyclic, we have from the periodicity of Tate's cohomology group

(4.2) $$\hat{H}^q(G_p/N, H^0(N, A^{\bullet})) = 0$$

for every integer $q$. But by (4.1) and (4.2), we can deduce that $\hat{H}^q(G, A^{\bullet}) = 0$ for every integer $q > 0$ and every subgroup $H$. Note that we use Proposition 1 and the assumption on $A^{\bullet}$.

Next we shall show $\hat{H}^0(G_p, A^{\bullet}) = 0$. Recall that we have assumed $A^q = 0$ for $q > 0$. By (4.1) and Proposition 1, for any $a \in H^0(G_p, A^{\bullet})$ there exists $b \in H^0(N, A^{\bullet})$ such that $\alpha = N_{G_p/N}(b)$. But from the assumption of induction we see that for any $b \in H^0(N, A^{\bullet})$ there is $c \in \mathscr{H}^0(A^{\bullet})$ such that $c = N_N(b)$. Thus we see that for any $a \in H^0(G_p, A^{\bullet})$ there exists $c \in \mathscr{H}^0(A^{\bullet})$ such that $a = N_{G_p}(c)$. Hence, by Proposition 1 we can deduce $\hat{H}^0(G_p, A^{\bullet}) = 0$.

Now we must prove that $\hat{H}^q(H, A^{\bullet}) = 0$ for every integer $q < 0$ and every subgroup $H$ of $G$. But this can be reduced to the case $\hat{H}^0(H, *)$ by Lemma 3.

For a general bounded complex $A^{\bullet}$, there is an integer $m$ such that for every $q > m$, $A^q = 0$. So we have only to prove that $A^{\bullet}[m]$ is cohomologically trivial. But this was already achieved above. **Q. E. D.**

**Remark.** In [2] this "twin number criterion" (see [2], §2, Lemma 2.2.) was proved for only two-term complexes of $G$-modules, which all terms are zero except 0-th and $(-1)$-th term. We have proved this criterion by using the hypercohomology spectral sequence, and the fact that the complexes considered are two-term. For more details and an alternative proof, see[2], §2, Lemma 2.2.

**Corollary 5.** *Let $A^{\bullet}$ and $B^{\bullet}$ be bounded complexes of $G$-modules and $f^{\bullet}$ be* *a morphism from the complex* $A^{\cdot}$ *to the complex* $B^{\cdot}$. *Assume that for every* $p$-*Sylow subgroup* $G_p$ *there is an integer* $i_p$ *such that*

$$f^{i_p} : \hat{H}^{i_p}(G_p, A^{\cdot}) \to \hat{H}^{i_p}(G_p, B^{\cdot})$$

*is surjective,*

$$f^{i_p+1} : \hat{H}^{i_p+1}(G_p, A^{\cdot}) \to \hat{H}^{i_p+1}(G_p, B^{\cdot})$$

*is bijective, and*

$$f^{i_p+2} : \hat{H}^{i_p+2}(G_p, A^{\cdot}) \to \hat{H}^{i_p+2}(G_p, B^{\cdot})$$

*is injective.*

*Then, for every integer* $q$ *and every subgroup* $H$ *of* $G$

$$f^q : \hat{H}^q(H, A^{\cdot}) \to \hat{H}^q(H, B^{\cdot})$$

*is an isomorphism.*

*Proof.* We have the following distinguished triangle:

$$A^{\cdot} \xrightarrow{f^{\cdot}} B^{\cdot} \to C(f^{\cdot}) \to A^{\cdot}[1],$$

where $C(f^{\cdot})$ is the mapping cone of $f^{\cdot}$.

From this triangle we have the following long exact sequence:

$$\begin{aligned} \cdots &\to \hat{H}^{i_p}(G_p, A^{\cdot}) \to \hat{H}^{i_p}(G_p, B^{\cdot}) \to \hat{H}^{i_p}(G_p, C(f^{\cdot})) \\ &\to \hat{H}^{i_p+1}(G_p, A^{\cdot}) \to \hat{H}^{i_p+1}(G_p, B^{\cdot}) \to \hat{H}^{i_p+1}(G_p, C(f^{\cdot})) \\ &\to \hat{H}^{i_p+2}(G_p, A^{\cdot}) \to \hat{H}^{i_p+2}(G_p, B^{\cdot}) \to \hat{H}^{i_p+2}(G_p, C(f^{\cdot})) \to \cdots, \end{aligned} \tag{5.1}$$

for every $p$. By the assumptions of the corollary and the above exact sequence, we see

$$\hat{H}^{i_p}(G_p, C(f^{\cdot})) = \hat{H}^{i_p+1}(G_p, C(f^{\cdot})) = 0$$

for every $p$. We can easily deduce $\hat{H}^q(H, C(f^{\cdot})) = 0$ for every integer $q$ and every subgroup $H$ of $G$ by Proposition 4. Noting exact sequence (5.1) and this fact, we have our assertion. **Q. E. D.**

The next theorem which seems to be a satisfactory generalization of the Tate-Nakayama theorem, is the main result of this paper.

**Theorem 6** (Generalized Tate-Nakayama theorem). *Let* $G$ *be a finite group,* $A^{\cdot}$ *a bounded complex of* $G$-*modules. Let* $a$ *be an element of* $\hat{H}^2(G, A^{\cdot})$. *Assume that for each* $p$-*Sylow subgroup* $G_p$ *of* $G$:

(1) $\hat{H}^1(G_p, A^{\cdot}) = 0$.

(2) $\hat{H}^2(G_p, A^{\cdot})$ *is generated by the element* $\mathrm{Res}_{G/G_p}(a)$ *whose order is equal to* $|G_p|$.

*Then, for every integer* $q$ *and every subgroup* $H$ *of* $G$, *we have*

$$\hat{H}^{q-2}(H, \boldsymbol{Z}) \simeq \hat{H}^q(H, A^{\cdot}).$$

*Proof.* By the cup-product pairing, we have the following bilinear map:

$$\hat{H}^q(H, \boldsymbol{Z}) \times \hat{H}^2(H, A^{\cdot}) \to \hat{H}^{q+2}(H, A^{\cdot}).$$

On the other hand, from Lemma 3, we can construct a complex $I^{\cdot}$ such that

$$\hat{H}^q(H, I^{\cdot}) \simeq \hat{H}^{q+2}(H, A^{\cdot}).$$

Since the above isomorphism is, as is well-known, the composition of connecting homomorphisms of cohomology, it is commutative with the cup-products. Therefore it is sufficient to show that $\hat{H}^q(H, \boldsymbol{Z}) \simeq \hat{H}^q(H, I^{\cdot})$.

Take an element $b \in \hat{H}^0(G_p, I^{\cdot})$ corresponding to the element $\mathrm{Res}_{G/G_p}(a)$, which induces the morphism of complexes from $\boldsymbol{Z}$ to $I^{\cdot}$ in an obvious fashion. From assumption (2) of the theorem, this morphism induces an isomorphism between $\hat{H}^0(G_p, \boldsymbol{Z})$ and $\hat{H}^0(G_p, I^\bullet)$. From assumption (1), we see that the morphism from $\hat{H}^{-1}(G_p, \boldsymbol{Z})$ to $\hat{H}^{-1}(G_p, I^\bullet)$ is surjective. Since $\hat{H}^1(G_p, \boldsymbol{Z}) = 0$, we can always deduce that the morphism from $\hat{H}^1(G_p, \boldsymbol{Z})$ to $\hat{H}^1(G_p, I^\bullet)$ is injective.

Noting Corollary 5, we have the theorem. **Q. E. D.**

## References

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