44. *Mountain Pass Theorems for Non-differentiable Functions and Applications*

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Abstract: We present some versions of the Mountain Pass Theorem of Ambrosetti and Rabinowitz for locally Lipschitz functionals. A multivalued elliptic problem is solved as an application of these results.

Key words: Clarke subdifferential; critical point theory; multivalued elliptic problem.

1. Introduction. The Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] is a very useful tool for finding critical points of C^1-functionals. We shall give some variants of this celebrated theorem for locally Lipschitz mappings.

Throughout, X will be a real Banach space. As usual, X^* denotes the dual of X and $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. We say that a function $f : X \rightarrow \boldsymbol{R}$ is locally Lipschitz ($f \in Lip_{loc}(X, \boldsymbol{R})$) if, for each $x \in X$, there is a neighbourhood V of x and a constant $k = k(V)$ depending on V such that $|f(y) - f(z)| \leq k \| y - z \|$ for each $y, z \in V$.

We recall in what follows the definition of the Clarke subdifferential and some of its most important properties (see, for details, [6]).

For each $x, v \in X$, we define the generalized directional derivative at x in the direction v of a given $f \in Lip_{loc}(X, \boldsymbol{R})$ as

$$f^0(x, v) = \limsup_{y \rightarrow x,\, \lambda \searrow 0} (f(y + \lambda v) - f(y)) / \lambda.$$

It is known that, if $f \in Lip_{loc}(X, \boldsymbol{R})$, then $f^0(x, v)$ is a finite number and $|f^0(x, v)| \leq k \| v \|$. The mapping $v \mapsto f^0(x, v)$ is positively homogeneous and subadditive, and then, it is convex continuous. The generalized gradient (the Clarke subdifferential) of f at x is the subset $\partial f(x)$ of X^* defined by $\partial f(x) = \{x^* \in X^* ; f^0(x, v) \geq \langle x^*, v\rangle,\ \forall\ v \in X\}$.

The fundamental properties of the Clarke subdifferential are: a) For each $x \in X$, $\partial f(x)$ is a nonempty convex ★-compact subset of X^*.

b) For each $x, v \in X$, we have $f^0(x, v) = \max\{\langle x^*, v\rangle ; x^* \in \partial f(x)\}$.

c) The set-valued mapping $x \rightarrow \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X$, $\varepsilon > 0$, $v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\| x - x_0 \| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $| \langle x^* - x_0^*, v\rangle | < \varepsilon$.

d) The function $f^0(\cdot,\cdot)$ is upper semi-continuous.

e) If f attains a local minimum or maximum at x, then $0 \in \partial f(x)$.

f) The function $\lambda(x) = \min\{\| x^* \| ; x^* \in \partial f(x)\}$ exists and is lower

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semi-continuous.

Definition 1. A point $u \in X$ is said to be a crilical point of $f \in Lip_{loc}(X, \boldsymbol{R})$ if $0 \in \partial f(u)$, namely $f^0(x, v) \geq 0$ for every $v \in X$. *A* real nymber c is called a critical value of f if there is a critical point $u \in X$ such that $\partial f(u) = c$.

Definition 2. If $f \in Lip_{loc}(X, \boldsymbol{R})$ and c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short $(PS)_c$) if any sequence (x_n) *in* X with the properties $lim_{n\to\infty} f(x_n) = c$ and $lim_{n\to\infty} \lambda(x_n) = 0$ has a convergent subsequence.

2. Main results. In what follows, f will be a locally Lipschitz function on the real Banach space X. Let K be a compact metric space and let K^* be a nonempty closed subset of K. If p^* is a fixed continuous map defined on K, let $\mathcal{P} = \{p \in C(K, X) \ ; \ p = p^* \text{ on } K^*\}$. Define

(1) $$c = inf_{p\in\mathcal{P}}\, max_{t\in K} f(p(t)).$$

Clearly, $c \geq max_{t\in K^*} f(p^*(t))$.

Theorem 1. *Assume that*

(2) $$c > max_{t\in K^*} f(p^*(t))$$

Then there exists a sequence (x_n) *in* X *such that*

i) $\lim_{n\to\infty} f(x_n) = c$, ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Corollary 1. *If* f *has* $(PS)_c$ *and satisfies the same assumptions as in Theorem* 1, *then* c *is a critical value of* f, *corresponding to a critical point which is not in* $p^*(K^*)$.

The proof follows from Theorem 1 and the lower-semicontinuity of the function λ.

Corollary 2. *Suppose* $f(0) = 0$ *and there exists* $v \in X \setminus \{0\}$ *such that* $f(v) \leq 0$. *If* $c > 0$ *and* f *satisfies* $(PS)_c$, *then* c *is a critical vàlue of* f.

For the proof, it suffices to apply Corollary 1 for $K = [0, 1]$, $K^* = \{0, 1\}$, $p^*(0) = 0$ and $p^*(1) = v$.

If (2) fails, a sufficient condition which ensures the validity of Theorem 1 is given by the following result, which is a variant of Theorem 1 in [9].

Theorem 2. *Assume that for every* $p \in \mathcal{P}$ *there is some point* $t \in K \setminus K^*$ *such that* $f(p(t)) \geq c$. *Then there exists a sequence* (x_n) *in* X *such that*

i) $\lim_{n\to\infty} f(x_n) = c$, ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Assume, in addition, that f *satisfies* $(PS)_c$. *Then* c *is a critical value of* f. *Furthermore, if* (p_n) *is any sequence in* $\mathcal{P}$ *such that* $lim_{n\to\infty} max f(p_n(t)) = c$, *then there exists a sequence* (t_n) *in* K *such that* $lim_{n\to\infty} f(p_n(t_n)) = c$ *and* $lim_{n\to\infty} \lambda(p_n(t_n)) = 0$.

Proof of Theorem 1. We apply Ekeland's variational principle to the functional $\psi(p) = max\,\{f(p(t)) \ ; \ t \in K\}$ defined on the complete metric space $\mathcal{P}$, endowed with the usual metric. The function ψ is continuous on $\mathcal{P}$ and bounded below, because $\psi(p) \geq max_{t\in K^*} f(p^*(t))$. Since $c = inf_{p\in\mathcal{P}}\, \psi(p)$, it follows that, for every $\varepsilon > 0$, there exists $p \in \mathcal{P}$ such that

(3) $$\psi(q) - \psi(p) + \varepsilon d(p, q) \geq 0, \text{ for each } q \in \mathcal{P}$$

(4) $$c \leq \psi(p) \leq c + \varepsilon.$$

Setting

$$B(p) = \{t \in K\ ; f(p(t)) = \phi(p)\}$$

it suffices to prove that there exists $t' \in B(p)$ such that

(5) $$\lambda(p(t')) \leq 2\varepsilon.$$

Then the conclusion of the theorem follows easily by choosing $\varepsilon = \frac{1}{n}$ and $x_n = p(t')$.

We need now the following result:

Lemma 1. *Let M be a compact metric space and let $\varphi : M \to 2^{X^*}$ be a set-valued mapping which is upper semi-continuous (in the sense of property c)) and with ★-compact convex values. For $t \in M$ denote $\gamma(t) = inf\{\| x^* \| ; x^* \in \varphi(t)\}$ and $\gamma = inf\{\gamma(t)\ ; t \in M\}$.*

Then, given $\varepsilon > 0$, there exists a continuous function $v : M \to X$ such that for all $t \in M$ and $x^* \in \varphi(t)$, $\| v(t) \| \leq 1$ and $\langle x^*, v(t) \rangle \geq \gamma - \varepsilon$.

For the proof of this lemma, see [5]. Applying Lemma 1 for $M = B(p)$ and $\varphi(t) = \partial f(p(t))$ we obtain a continuous function $v : B(p) \to X$ such that for all $t \in B(p)$ and $x^* \in \partial f(p(t))$,

(6) $$\| v(t) \| \leq 1 \text{ and } \langle x^*, v(t) \rangle \geq \gamma - \varepsilon.$$

where $\gamma = inf_{t \in B(p)} \lambda(p(t))$.

It follows that for each $t \in B(p)$,

$$f^0(p(t), -v(t)) = max\{< x^*, -v(t) > ; x^* \in \partial f(p(t))\} =$$
$$= -min\{< x^*, v(t) > ; x^* \in \partial f(p(t))\} \leq -\gamma + \varepsilon.$$

By assumption (2), $B(p) \cap K^* = \emptyset$. Thus there is a continuous function $w : K \to X$ which extends v such that $w|_{K^*} = 0$ and $\| w(t) \| \leq 1$ for each $t \in K$. We take for q, in (3), small variations of the path p: $q_h(t) = p(t) - hw(t)$ where $h > 0$ is small enough.

It follows from (3) that for every $h > 0$

(7) $$-\varepsilon \leq -\varepsilon \| w \|_\infty \leq \frac{\phi(q_h) - \phi(p)}{h}.$$

In what follows, $\varepsilon > 0$ is fixed while we let $h \to 0$. Let $t_h \in K$ be such that $f(q_h(t_h)) = max_{t \in K} f(q_h(t))$. For a suitable sequence $h_n \to 0$, t_{h_n} converges to some t_0 which belongs to $B(p)$. Therefore,

$$\frac{\phi(q_h) - \phi(p)}{h} = \frac{\phi(p - hw) - \phi(p)}{h} \leq \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}.$$

It follows from (7) that

$$\varepsilon \leq \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h} \leq$$

$$\leq \frac{f(p(t_h) - hw(t_0)) - f(p(t_h))}{h} + \frac{f(p(t_h) - hw(t_h)) - f(p(t_h) - hw(t_0))}{h}.$$

Using the fact that f is locally Lipschitz and that the sequence (t_{h_n}) converges to t_0, we get

$$\lim_{n\to\infty} \frac{f(p(t_{h_n}) - h_n w(t_{h_n})) - f(p(t_{h_n}) - h_n w(t_0))}{h_n} = 0.$$

Therefore,

$$-\varepsilon \leq \limsup_{n\to\infty} \frac{f(p(t_0) + z_n - h_n w(t_0)) - f(p(t_0) + z_n)}{h_n}$$

where $z_n = p(t_{h_n}) - p(t_0)$. Consequently,

$$-\varepsilon \leq f^0(p(t_0), -w(t_0)) = f^0(p(t_0), -v(t_0)) \leq -\gamma + \varepsilon$$

which implies $\gamma = \inf\{\|x^*\|;\ x^* \in \partial f(p(t)),\ t \in B(p)\} \leq 2\varepsilon$.

It follows from the lower semi-continuity of λ that there is some $t' \in B(p)$ such that $\lambda(p(t')) = \inf\{\|x^*\|;\ x^* \in \partial f(p(t))\} \leq 2\varepsilon$.

Proof of Theorem 2. We shall apply Ekeland's variational principle to the functional $\psi_\varepsilon(p) = \max\{f(p(t)) + \varepsilon d(t)\ ;\ t \in K\}$
for each $\varepsilon > 0$ and $p \in \mathcal{P}$, where $d(t) = \min\{dist(t, K^*), 1\}$.

If $c_\varepsilon = \inf_{p\in\mathcal{P}} \psi_\varepsilon(p)$, then $c \leq c_\varepsilon \leq c + \varepsilon$.

Applying Ekeland's variational principle, we get a path $p \in \mathcal{P}$ such that for each $q \in \mathcal{P}$,

(8) $$\psi_\varepsilon(q) - \psi_\varepsilon(p) + \varepsilon d(p, q) \geq 0$$

(9) $$c \leq c_\varepsilon \leq \psi_\varepsilon(p) \leq c_\varepsilon + \varepsilon \leq c + 2\varepsilon.$$

Setting $B_\varepsilon(p) = \{t \in K\ ; f(p(t)) + \varepsilon d(t) = \psi_\varepsilon(p)\}$,
it remains to prove that we can find some $t' \in B_\varepsilon(p)$ such that $\lambda(p(t')) \leq 2\varepsilon$. We obtain thereafter the conclusion of the first part of the theorem by choosing $\varepsilon = \frac{1}{n}$ and $x_n = p(t')$.

Applying Lemma 1 for $M = B_\varepsilon(p)$ and $\varphi(t) = \partial f(p(t))$, we get a continuous map $v : B_\varepsilon(p) \to X$ such that for all $t \in B_\varepsilon(p)$ and $x^* \in \partial f(p(t))$,

$$\|v(t)\| \leq 1 \text{ and } < x^*, v(t) > \geq \gamma - \varepsilon$$

where $\gamma = \inf\{\lambda(p(t))\ ;\ t \in B_\varepsilon(p)\}$.

But our hypothesis implies

(10) $$\psi_\varepsilon(p) > \max\{f((p(t))\ ;\ t \in K^*\}.$$

Hence, $B_\varepsilon(p) \cap K^* = \emptyset$. Thus, there exists a continuous function w defined on K which extends v such that $w_{|B_\varepsilon(p)} = 0$ and $\|w(t)\| \leq 1$ for all $t \in K^*$. We take for q, in (8), small variations of the path p:
$q_h(t) = p(t) - hw(t)$ for $h > 0$ small enough.

In what follows, $\varepsilon > 0$ will be fixed while we let $h \to 0$. There exists $t_h \in B_\varepsilon(p)$ such that $f(q(t_h)) + \varepsilon d(t_h) = \psi_\varepsilon(q_h)$. For a suitable sequence $h_n \to 0$, t_{h_n} converges to some $t_0 \in B_\varepsilon(p)$. It follows that

$$-\varepsilon \leq -\varepsilon\|w\|_\infty \leq \frac{\psi_\varepsilon(q_h) - \psi_\varepsilon(p)}{h} = \frac{f(q_h(t_h)) + \varepsilon d(t_h) - \psi_\varepsilon(p)}{h} \leq$$

$$\leq \frac{f(q_h(t_h)) - f(p(t_h))}{h} = \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}.$$

With the same reasoning as in the proof of Theorem 1 we get that there is some $t' \in B_\varepsilon(p)$ such that $\lambda(p(t')) \leq 2\varepsilon$.

If f has $(PS)_c$, then c is a critical value because of the lower semi-continuity of λ.

For the second part of the proof, there exists, by Ekeland's varational principle, a sequence of paths (q_n) in $\mathcal{P}$ such that for each $q \in \mathcal{P}$,

$\psi_{\varepsilon_n^2}(q) - \psi_{\varepsilon_n^2}(q_n) + \varepsilon_n d(q, q_n) \geq 0$ and $\psi_{\varepsilon_n^2}(q_n) \leq \psi_{\varepsilon_n^2}(p_n) - \varepsilon_n d(p_n, q_n)$,
where (ε_n) is a sequence of positive numbers which tends to 0 and (p_n) are

paths in $\mathscr{P}$ such that $\psi_{\varepsilon_n^2}(p_n) \le c + 2\varepsilon_n^2$. It follows that $d(p_n, q_n) \le 2\varepsilon_n$. Applying the preceding argument to (q_n), instead of p, we find some elements $t_n \in K$ such that $c - \varepsilon_n^2 \le f(q_n(t_n)) \le c + 2\varepsilon_n^2$ and $\lambda(q_n(t_n)) \le 2\varepsilon_n$.

This is the desired sequence (t_n). Indeed, by $(PS)_c$, a subsequence of $q_n(t_n)$ converges to a critical point and then the corresponding subsequence of $p_n(t_n)$ converges to the same limit. A standard argument, using the contunuity of f and the lower semi-continuity of λ, shows that for the full sequence, $lim_{n\to\infty} f(p_n(t_n)) = c$ and $lim_{n\to\infty} \lambda(p_n(t_n)) = 0$.

3. An application. Let Ω be a smooth bounded domain in $\boldsymbol{R}^N (N \ge 3)$ and g be a measurable function defined on $\Omega \times \boldsymbol{R}$ satisfying, for all $(x, t) \in \Omega \times \boldsymbol{R}$

(11) $$|g(x, t)| \le C_0(1 + |t|^p)$$

where C_0 is a positive constant and $1 \le p < \frac{N+2}{N-2}$.

Define the functional ψ in $L^{p+1}(\Omega)$ by

$$\psi(u) = \int_\Omega \int_0^{u(x)} g(x, t)\,dtdx.$$

The fact that ψ is a locally Lipschitz function in $L^{p+1}(\Omega)$ follows from the growth condition (11) and the Hölder inequality.

Setting $G(x, t) = \int_0^t g(x, s)\,ds$, then, by Theorem 2.1. in [Ch], the Clarke subdifferential $\partial_t G(x, t)$ of the mapping $t \mapsto G(x, t)$ is given by $\partial_t G(x, t) = [\underline{g}(x, t), \bar{g}(x, t)]$, where

$$\underline{g}(x, t) = lim_{\varepsilon\searrow 0}\, ess\ inf[g(x, s)\ ;\ |t - s| < \varepsilon\}$$
$$\bar{g}(x, t) = lim_{\varepsilon\searrow 0}\, ess\ sup[g(x, s)\ ;\ |t - s| < \varepsilon\}.$$

Assuming that

(12) $\underline{g}$ and $\bar{g}$ are measurable on $\Omega \times \boldsymbol{R}$,

by Theorems 2.1. and 2.2. in [Ch] it follows that

(13) $$\partial\psi_{|H_0^1(\Omega)}(u) \subset \partial\psi(u) \subset \partial_t G(x, t) \quad a.e.\ x \in \Omega.$$

We suppose, in addition, that

(14) $g(x, 0) = 0$ and $lim\ sup_{t\to 0}\left|\frac{g(x, t)}{t}\right| < \lambda_1$ uniformly in $x \in \Omega$

and

(15) $$\mu G(x, t) \le \begin{cases} t\underline{g}(x, t),\ a.e.\ x \in \Omega,\ t \ge r \\ t\bar{g}(x, t),\ a.e.\ x \in \Omega,\ t \le -r \end{cases}$$

for some $\mu > 2$ and $r \ge 0$.

Proposition 1. *Let* $a \in L^\infty(\Omega)$ *be a non-negative function and suppose that conditions* (11), (12), (13), (14) and (15) *hold. Then the multivalued nonlinear elliptic problem*

(16) $$-\Delta u(x) + a(x)u(x) \in [\underline{g}(x, u(x)), \bar{g}(x, u(x))] \quad a,e.\ x \in \Omega$$

has a solution in $H_0^1(\Omega) \cap W^{2,p'}(\Omega)$, *where* p' *is the conjugated exponent of* p.

Sketch of the proof. We consider in $H_0^1(\Omega)$ the locally Lipschitz function

$$\varphi(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega a(x)u^2(x)dx - \psi(u).$$

To prove our statement it suffices to show that φ has a critical point

$u_0 \in H_0^1(\Omega)$ corresponding to a positive critical value. Indeed, it is obvious that $\partial\varphi(u) = -\Delta u + a(x)u - \partial\psi_{|H_0^1(\Omega)}(u)$ in $H^{-1}(\Omega)$.

If u_0 would be a critical point of φ it follows that there would exist $w \in \partial\varphi_{H_0^1(\Omega)}(u_0)$ such that $-\Delta u_0 + a(x)u_0 = w$ in $H^{-1}(\Omega)$.
But $w \in L^{p'}(\Omega)$. Then by a standard regularity theorem for elliptic equations we obtain that $u_0 \in W^{2,p}(\Omega)$ and u_0 is a solution of the problem (16).

To prove that φ has a critical point we apply Corollary 2, by showing that φ satisfies the Palais-Smale condition and the following geometrical assumptions:

$\varphi(0) = 0$ and there exists $v \in H_0^1(\Omega)$ such that $\varphi(v) \leq 0$.

There exist $c > 0$ and $0 < R < \| v \|$ such that $\varphi_{|\partial B(0,R)} \geq c$.

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