39. *On the Measure on the Set of Positive Integers*

By Kōsaku OKUTSU

Department of Mathematics, Gakushuin University

(Communicated by Shokichi IYANAGA, M. J. A., June 8, 1993)

R. C. Buck [1] constructed as follows a measure μ on the set $\boldsymbol{N}=\{0,1,2,\ldots\}$ of positive integers. For an arithmetic progression $A=\{an+b \mid n \in \boldsymbol{N}\}=a\boldsymbol{N}+b$, $a, b \in \boldsymbol{N}$, $a \neq 0$, put $\mu(A)=a^{-1}$. Let $\boldsymbol{A}$ be the class of all arithmetic progressions and $\boldsymbol{B}$ the class of subsets of $\boldsymbol{N}$ which are finite disjoint unions of elements of $\boldsymbol{A}$; thus if $B \in \boldsymbol{B}$, then $B=\sum_{i=1}^{k} A_i$ (disjoint union) $A_i \in \boldsymbol{A}$, $i=1,2,\ldots,k$. For such B, put $\mu(B)=\sum_{i=1}^{k} \mu(A_i)$. For any subset C of $\boldsymbol{N}$, $\mu(C)$ is defined to be inf $\mu(B)$, $B \in \boldsymbol{B}$ and $C \subset B \cup F$, where F is a finite subset of $\boldsymbol{N}$.

On the other hand, we have another measure ν on $\boldsymbol{N}$, used by J.-L. Mauclaire [2] to obtain various results. Let P denote the set of all prime numbers. For $p \in P$, the additive group $\boldsymbol{Z}_p$ of p-adic integers with p-adic topology is a compact abelian group, which has therefore the Haar measure ν_p with $\nu_p(\boldsymbol{Z}_p)=1$. The product group $G=\Pi_{p\in P}\boldsymbol{Z}_p$ with the product topology is again a compact abelian group with product measure $\nu=\Pi_{p\in P}\nu_p$. $\boldsymbol{Z}$ is considered as a dense subgroup in G, and $\boldsymbol{N}$ as an open and closed subset of $\boldsymbol{Z}$ which is also dense in G.

J.-L. Mauclaire [3] discussed the relationship between μ and ν using Riemann-Stieltjes integration. In this note, we shall show that this relationship can be directly clarified using only topological considerations.

Remark. The above introduced notations $\boldsymbol{A}$, $\boldsymbol{B}$, μ, ν, ν_p will be used throughout this note in the same meanings. Let us recall that $U_p(x, e)=x+p^e\boldsymbol{Z}_p$, $x \in \boldsymbol{Z}_p$, $e \in \boldsymbol{N}$, constitute an open basis of $\boldsymbol{Z}_p$ and $V_S(U_p(x_p, e_p))=\Pi_{p\in S}U_p(x_p, e_p)\times\Pi_{q\in P-S}\boldsymbol{Z}_q$ where S runs over the finite subset of P, $x_p \in \boldsymbol{Z}_p$, $e_p \in \boldsymbol{N}$, an open basis of G. For a subset M of G, $\overline{M}$ will denote the closure of M in G. Recall, furthermore, that $\nu_p(U_p(x, e))=\nu_p(x+p^e\boldsymbol{Z}_p)$ dose not depend on x and is equal to p^{-e}, so that $\nu(V_S(U_p(x_p, e_p)))=\Pi_{p\in S}p^{-e_p}$.

Our main result will follow from the following two propositions.

Proposition 1. *For any open and closed non-empty subset O in G, $O \cap \boldsymbol{N}$ belongs to $\boldsymbol{B}$.*

Proof. O is a union of sets of form $V_S(U_p(x_p, e_p))$, because O is open. As G is compact, O is also compact. So O is a finite union of $V_S(U_p(x_p, e_p))$. Now $V_S(U_p(x_p, e_p)) \cap \boldsymbol{N}=a\boldsymbol{N}+b \in \boldsymbol{A}$ where $a=\Pi_{p\in P}p^{e_p}$ and $b \equiv x_p \bmod p^{e_p}$, so that $O \cap \boldsymbol{N}$ is an element of $\boldsymbol{B}$.

Proposition 2. *For $B \in \boldsymbol{B}$, $\overline{B}$ is an open and closed subset of G, and $\mu(B)=\nu(\overline{B})$.*

Proof. For $A=a\boldsymbol{N}+b \in \boldsymbol{A}$, we set $a=\Pi_{p\in P}p^{e_p}$ where S is a finite subset of $\boldsymbol{P}$. Then we have $\overline{A} = a\overline{\boldsymbol{N}} + \overline{b} = aG + b = \Pi_{p\in S} U_p(\overline{b}, e_p) \times \Pi_{q\in P-S} \boldsymbol{Z}_q = V_S(U_p(b, e_p))$. And we have $\nu(\overline{A}) = a^{-1}$. So $\mu(A) = \nu(\overline{A})$. Now set $B = \sum_{i=1}^{k} A_i$ (disjoint union) $A_i \in \boldsymbol{A}$, $i = 1,2,\ldots,k$. Then $\overline{B} = \cup_{i=1}^{k} \overline{A}_i$. As $\overline{A}_i \cap \boldsymbol{N} = A_i$, we have $\overline{A}_i \cap \overline{A}_j \cap \boldsymbol{N} = \emptyset$ when $i \neq j$. As $\overline{A}_i \cap \overline{A}_j$ is open and closed, we have $\overline{A}_i \cap \overline{A}_j = \emptyset$ when $i \neq j$. Thus $\nu(\overline{B}) = \sum_{i=1}^{k} \nu(\overline{A}_i) = \sum_{i=1}^{k} \mu(A_i) = \mu(B)$.

Theorem. *For any subset C of $\boldsymbol{N}$, we have $\mu(C) = \nu(\overline{C})$.*

Proof. Because $\nu(\overline{C}) = \inf \nu(U)$ where U is open subset of G such that $\overline{C} \subset U \cup F$ for some finite set F, and $\overline{C}$ is compact, we have $\nu(\overline{C}) = \inf \nu(O)$ where O is open and closed subset of G such that $\overline{C} \subset O \cup F$ for some finite set F. As $\nu(O) = \mu(O \cap \boldsymbol{N})$, and as $(O \cap \boldsymbol{N}) \in \boldsymbol{B}$, we have $\nu(\overline{C}) = \inf \nu(\overline{B})$, $B \in \boldsymbol{B}$ and $C \subset B \cup E$ for some finite subset E of $\boldsymbol{N}$. Thus we have $\nu(\overline{C}) = \mu(C)$.

References

[1] R. C. Buck: The measure theoretic approach to density. Amer. J. Math., **68**, 560–580 (1946).

[2] J.-L. Mauclaire: Intégration et théorie des nombres. Hermann (1986).

[3] ——: Suites limite-périodiques et théorie des nombres. VI. Proc. Japan Acad., **57A**, 223–225 (1981).