19. A Note on Untwisted Deform-spun 2-knots

By Masakazu TERAGAITO
Department of Mathematics, Faculty of Science, Kobe University

(Communicated by Heisuke HIRONAKA, M. J. A., April 13, 1992)

In [5] Litherland introduced the process of deform-spinning of which twist-spinning [8], roll-spinning [1] are particular examples. Given a 1-knot (S^3, K), let g be a self-homeomorphism of (S^3, K) with $g=id$ on a tubular neighbourhood $K\times D^2$ of K. The deform-spun 2-knot corresponding to g is defined as follows.

Fix a point z on K. Take a ball neighbourhood K_- of z in K, and set $B_-=K_-\times D^2$. Let (B_+, K_+) be the complementary ball pair of (B_-, K_-) which is the standard ball pair. Then we construct $\partial(B_+, K_+)\times B^2\cup_\partial(B_+, K_+)\times_g\partial B^2$, where

$$(B_+, K_+)\times_g\partial B^2=(B_+, K_+)\times I/((x, 0)\sim(g(x), 1) \text{ for all } x\in B_+).$$

This is a locally-flat sphere pair depending only on the isotopy class γ of g (rel $K\times D^2$). (See [5].) We denote this 2-knot by $(S^4, \gamma K)$, and call it the *deform-spun knot* of K corresponding to γ, or g.

Let $\mathcal{H}(K)$ be the group of self-homeomorphisms g of (S^3, K) with $g=id$ on $K\times D^2$ and let $\mathcal{D}(K)$ be $\mathcal{H}(K)$ modulo isotopy rel $K\times D^2$. We call elements of $\mathcal{D}(K)$ *deformations* of K. It is well-known ([4], [7]) that the exterior $X(K)=\mathrm{cl}(S^3-K\times D^2)$ admits a map $p: X(K)\to\partial D^2$ such that $p|_{\partial X(K)}: \partial X(K)=K\times\partial D^2\to\partial D^2$ is the projection. We will refer to such a map as a *projection* for K. (We always assume that $K\times\theta$ is null-homologous in $X(K)$ for $\theta\in\partial D^2$.) A deformation $\gamma\in\mathcal{D}(K)$ is said to be *untwisted* if there is a projection p for K and a representative g of γ with $p(g|_{X(K)})=p$. If γ is untwisted, we say that γK is untwisted.

For any 1-knot K, twist-spinning $\tau\in\mathcal{D}(K)$ can be defined. (See [5].) Zeeman showed that any ±1-twist-spun knot $\tau^{\pm1}K$ of K is unknotted [8]. But the deformation τ is *not* untwisted.

In this note we prove:

Theorem. *There exist infinitely many* 1*-knots K and untwisted deformations γ of K such that the corresponding untwisted deform-spun* 2*-knots γK are unknotted.*

Proof of Theorem. For a projection $p: X(K)\to\partial D^2$, if $\theta\in\partial D^2$ is a regular value, then $F^\theta=p^{-1}(\theta)$ is a compact, codimension 1 submanifold of $X(K)$ and $\partial F^\theta=K\times\{\theta\}$. That is, F^θ is a *Seifert surface* for K. (See [4], [7].) Let $\gamma\in\mathcal{D}(K)$ be an untwisted deformation and let g be a representative of γ with $p(g|_{X(K)})=p$. Then $g(F^\theta)=F^\theta$ for each $\theta\in\partial D^2$. A tubular neighbourhood of γK is $\partial K_+\times D^2\times B^2\cup K_+\times D^2\times\partial B^2$ and so γK has the exterior

$$X(\gamma K)=K_{-}\times\partial D^2\times B^2\cup X(K)\times_g \partial B^2.$$

The space $K_{-}\times\{\theta\}\times B^2\cup F^\theta\times_g \partial B^2$ gives a Seifert (hyper) surface for γK, which is denoted by γF^θ.

Lemma. *Let (S^3, K_i) be a 1-knot with projection $p_i: X(K_i)\to\partial D^2$, $i=1,2$. Let $F_i=p_i^{-1}(\theta)$ be a Seifert surface for K_i. Let $\gamma_i\in\mathcal{D}(K_i)$ be an untwisted deformation and let g_i be a representative with $p_i(g_i|_{X(K_i)})=p_i$. If there exists a homeomorphism $h: F_1\to F_2$ such that $hg_1=g_2h$, then untwisted deform-spun 2-knots γ_1K_1 and γ_2K_2 have homeomorphic Seifert (hyper) surfaces γ_1F_1 and γ_2F_2.*

The proof is straightforward, so we omit it.

We will denote the knot in Fig. 1 by $K(m,n)$, where $n\geq 3$ is an odd integer, and $2m+1$ indicates the number of half-twists (left-handed if $m\geq 0$, right-handed if $m<0$). Note that $K(0,n)$ and $K(-1,n)$ are torus knots of type $(2,n)$ and $(2,-n)$, respectively.

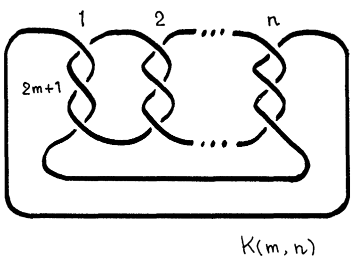

Fig. 1

We see that $K=K(m,n)$ has two periods, n and 2. That is, there are orientation-preserving self-homeomorphisms g_1 and g_2 of (S^3, K) such that the set J_i of fixed points of g_i is a 1-sphere disjoint from K, and g_1 and g_2 are of period n and 2, respectively. We may assume that J_1, J_2 are oriented so that $lk(K, J_1)=2$, $lk(K, J_2)=(-1)^m n$, $lk(J_1, J_2)=1$. Furthermore we assume that g_1 corresponds to the rotation through $2\pi/n$ around the axis J_1.

We will define an untwisted deformation of K using g_1 and g_2.

Let $q: S^3\to S^3/g_1g_2$ be the quotient map and let $\bar{K}=q(K)$, $\bar{J}_i=q(J_i)$. The map q is the $Z_n\oplus Z_2$-branched cover branched over $\bar{J}_1\cup\bar{J}_2$ corresponding to $Ker[\pi_1(S^3-\bar{J}_1\cup\bar{J}_2)\to H_1(S^3-\bar{J}_1\cup\bar{J}_2)\to Z_n\oplus Z_2]$, where the first map is the Hurewicz homomorphism and the second sends a meridian t_1 (t_2 resp.) of $\bar{J}_1$ ($\bar{J}_2$ resp.) to $(1,0)$ ($(0,1)$ resp.) $\in Z_n\oplus Z_2$. Let $\bar{p}: X(\bar{K})\to\partial D^2$ be a projection for $\bar{K}$, where a tubular neighbourhood $\bar{K}\times D^2$ of $\bar{K}$ is taken to be disjoint from $\bar{J}_i$. Then $q^{-1}(\bar{K}\times D^2)$ is a g_i-invariant tubular neighbourhood $K\times D^2$ of K such that $q(x,v)=(2nx, v)$ for $x\in K$, $v\in D^2$. Here, a circle is identified with the quotient space R/Z. We see that $g_1g_2|_{K\times D^2}$ is given by $(x,v)\to(x+1/2n, v)$. Take a g_i-invariant collar $\partial X(K)\times I$ of $\partial X(K)$ in $X(K)$ such that $\partial X(K)$ is identified with $\partial X(K)\times\{0\}$, and define a self-homeomorphism h of (S^3, K) by

$$h(x,\theta,\phi)=(x-(1-\phi)/2n,\theta,\phi) \quad \text{for } (x,\theta,\phi)\in K\times\partial D^2\times I,$$
$$h(x,v)=(x-1/2n,v) \quad \text{for } (x,v)\in K\times D^2,$$
$$h(y)=y \quad \text{for } y\in X(K)-\partial X(K)\times I.$$

Then $hg_1g_2|_{K\times D^2}=id$, $hg_1g_2|_{\mathrm{cl}(X(K)-\partial X(K)\times I)}=g_1g_2$, and $\bar{p}q(hg_1g_2|_{X(K)})=\bar{p}q$. Let ω be the class of hg_1g_2 in $\mathcal{D}(K)$. It is now evident that ω is untwisted with respect to a projection $\bar{p}q$ for K.

As shown in Fig. 2, $K(m,n)$ has a Seifert surface $F(m,n)$ of genus $(n-1)/2$, which is invariant under g_i and $J_1\cap F=\{2 \text{ points}\}$, $J_2\cap F=\{n \text{ points}\}$. Note that $F(0,n)$ and $F(-1,n)$ are fiber surfaces for $K(0,n)$ $K(-1,n)$, respectively.

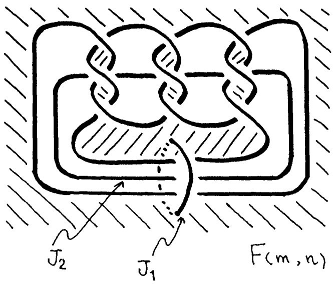

Fig. 2

Proof of Theorem. By Lemma, $\omega K(m,n)$ and $\omega K(0,n)$ have homeomorphic Seifert surfaces. The map hg_1g_2 is just the monodromy map on the fiber surface $F(0,n)$ (cf. [6: §9], [3: Chapter 19]). It follows that $\omega F(0,n)$ is a 3-cell. This completes the proof.

Remarks. (1) Moreover, we can prove that for any integer $r\geq 2$ the untwisted deform-spun 2-knot $\omega^r K(m,n)$ has a Seifert surface homeomorphic to the punctured Brieskorn 3-manifold $\Sigma(2,n,r)^\circ$. The r-fold cyclic branched covering of the $(2,n)$-torus knot is $\Sigma(2,n,r)$. Hence the r-twist-spun knot of the $(2,n)$-torus knot has a fiber $\Sigma(2,n,r)^\circ$. The knot $K(m,n)$ is a torus knot if and only if $m=0,-1$. We might expect that any nontrivial untwisted deform-spun 2-knot $\omega^r K(m,n)$ is non-fibered unless $K(m,n)$ is a torus knot. But I have been unable to prove this. In fact, Kanenobu [2] has observed that if $K(m,n)$ is not a torus knot and if $n\nmid m$ then $\omega^2 K(m,n)$ is non-fibered with Seifert surface $\Sigma(2,n,2)^\circ=L(n,1)^\circ$, the punctured lens space of type $(n,1)$.

(2) If $K(m,n)$ is a torus knot, then $\omega=\tau$ in $\mathcal{D}(K)$ [5: Cor. 6.5]. But if $K(m,n)$ is not a torus knot, the untwisted deformation ω is not contained in the subgroup $\langle\tau\rangle$ of $\mathcal{D}(K)$ generated by τ [5: Cor. 6.3].

References

[1] R. H. Fox: Rolling. Bull. Amer. Math. Soc., **72**, 162–164 (1966).

[2] T. Kanenobu: Untwisted deform-spun knots: Examples of symmetry-spun 2-knots. Transformation Groups (ed. K. Kawakubo). Lect. Notes in Math., vol. 1375, Springer-Verlag, New York, pp. 145–167 (1990).

[3] L. H. Kauffman: On knots. Annals of Math. Studies, vol. 115, Princeton Univ. Press, Princeton (1987).

[4] M. Kervaire and C. Weber: A survey of multidimensional knots. Lect. Notes in Math., vol. 685, Springer-Verlag, pp. 61–134 (1978).

[5] R. A. Litherland: Deforming twist-spun knots. Trans. Amer. Math. Soc., **250**, 311–331 (1979).

[6] J. W. Milnor: Singular points of complex hypersurfaces. Annals of Math. Studies, **61**, Princeton Univ. Press, Princeton (1968).

[7] D. Rolfsen: Knots and Links. Math. Lecture Series 7, Publish or Perish Inc., Berkeley (1976).

[8] E. C. Zeeman: Twisting spun knots. Trans. Amer. Math. Soc., **115**, 471–495 (1965).