16. *On the Ideal Class Groups of the p-Class Fields of Quadratic Number Fields*

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1. We fix an odd prime p. Let k be a quadratic number field and $\tilde{k}$ the Hilbert p-class field of k. Denote the p-primary parts of the ideal class groups of k and of $\tilde{k}$ by $\mathrm{Cl}^{(p)}(k)$ and by $\mathrm{Cl}^{(p)}(\tilde{k})$, respectively.

If the p-rank of $\mathrm{Cl}^{(p)}(k)$ is less than or equal to one, $\mathrm{Cl}^{(p)}(\tilde{k})$ is trivial. In fact, $\mathrm{Gal}(\tilde{k}/k)$ is then cyclic, and does not have any essential central extensions because the Schur multiplier of it is trivial.

If the p-rank of $\mathrm{Cl}^{(p)}(k)$ is greater than one, however, $\mathrm{Cl}^{(p)}(\tilde{k})$ is not trivial anymore. We see by Nomura [4] that $\tilde{k}/k$ has a non-trivial unramified central extension; in fact, we can show the following theorem by mathematical induction with Theorem 1 of [4]:

Theorem 1. *Suppose that the p-rank r of* $\mathrm{Cl}^{(p)}(k)$ *of a quadratic number field k is greater than one. Then* $\tilde{k}/k$ *has an unramified central extension* $K/\tilde{k}/k$ *whose group* $\mathrm{Gal}(K/k)$ *is isomorphic to the metabelian group D,*

$$D=\langle a_i, c_{i,j} \mid i=1, \cdots, r,\ j=i+1, \cdots, r\rangle,\quad a_i^{\varepsilon(i)}=c_{i,j}^{\varepsilon(i)}=1,\quad [a_i, a_j]=c_{i,j},$$
$$[a_i, c_{m,n}]=[c_{i,j}, c_{m,n}]=1,\quad i=1, \cdots, r,\quad j=i+1, \cdots, r,\quad 1\leq m<n\leq r,$$

where the abelian group $\mathrm{Cl}^{(p)}(k)$ *is of type* $(\varepsilon(1), \cdots, \varepsilon(r))$, $e(i)=p^{e_i}$, $i=1, \cdots, r$, $1\leq e_1\leq \cdots \leq e_r$. *In particular, we have* $|\mathrm{Cl}^{(p)}(\tilde{k})|\geq \prod_{i=1}^{r}\varepsilon(i)^{(r-i)}$ *and p-rank* $(\mathrm{Cl}^{(p)}(\tilde{k}))\geq\binom{r}{2}$.

For simplicity, put $C:=\mathrm{Cl}^{(p)}(k)$ and $G:=\mathrm{Gal}(\hat{k}/k)$ where $\hat{k}$ is the Hilbert p-clase field of $\tilde{k}$; denote the alternative product of C by itself by $C\wedge C$, and the lower central series of G by

$$G_1=G\supset G_2=[G_1, G]\supset G_3=[G_2, G]\supset \cdots.$$

Then $C\wedge C$ may be identified with the Schur multiplier of C, and is isomorphic to the commutator group

$$[D, D]=\langle c_{i,j} \mid 1\leq i<j\leq r\rangle$$

of D of the theorem. Since $[D, D]$ is contained in the center of D, we see

Corollary. *Let the notation and the assumptions be as above. Then the field K of the theorem is the maximal unramified central extension of* $\tilde{k}/k$; *hence, in particular,* G/G_3 *is isomorphic to the group D of the theorem, and* G_2/G_3 *is to* $C\wedge C$.

It is possible to give a better estimate of the size of $\mathrm{Cl}^{(p)}(\tilde{k})$ than that of Theorem 1 in case of an imaginary quadratic number field k; in fact, k

has a specific characteristic on capitulation of its ideals which claims a strong condition on the structure of G. We shall explain it in the next section. We see then not only that G itself can not be so small as to be isomorphic to the group D of Theorem 1 but also that the p-rank of $\mathrm{Cl}^{(p)}(\tilde{k})$ is much greater than $\binom{r}{2}$. Since $\hat{k}$ is a Galois extension of the rational number field $\boldsymbol{Q}$, there exists an element of order 2 in $\mathrm{Gal}(\hat{k}/\boldsymbol{Q})$ which induces a non-trivial automorphism of k over $\boldsymbol{Q}$; it gives an inner automorphism φ of order 2 which is non-trivial on $\mathrm{Gal}(\hat{k}/k)$. Our main purpose of this paper is to show

Theorem 2. *Let the notation and the assumptions be as above and suppose that k is an imaginary quadratic number field. Then we have*

(1) $|\mathrm{Cl}^{(p)}(\tilde{k})|=|C\wedge C|\cdot|G_3|=\{\prod_{i=1}^{r}\varepsilon(i)^{(r-i)}\}\cdot|G_3|$;

(2) $|G_3|\geq\prod_{i=1}^{r}[C:C^{\varepsilon(i)}]/\varepsilon(i)=|C\wedge C|^2$;

(3) $$p\text{-rank}\,(\mathrm{Cl}^{(p)}(\tilde{k}))\geq p\text{-rank}(C\wedge C)+p\text{-rank}(G_3^{1-\varphi})$$
$$\geq\binom{r}{2}+\binom{r+1}{2}-1=r^2-1;$$

(4) $p\text{-rank}(G_3^{1-\varphi})\geq\sum_{i=1}^{r}(r-\max\{n\,|\,e_1+\cdots+e_n\leq e_i\})\geq\binom{r+1}{2}-1.$

2. We denote $\mathrm{Gal}(\hat{k}/k)$ and $\mathrm{Gal}(\tilde{k}/k)$, simply by G and by A, respectively; the commutator group G_2 of G is equal to $\mathrm{Gal}(\hat{k}/\tilde{k})$; A is isomorphic to G/G_2. By class field theory, the Artin maps of k and of $\tilde{k}$ give isomorphisms of A and of G_2, respectively, onto $C=\mathrm{Cl}^{(p)}(k)$ and onto $\mathrm{Cl}^{(p)}(\tilde{k})$.

In our recent work [3], we see that the metabelian p-group G for an imaginary quadratic number field k satisfies the following two conditions (A) and (B):

(A) For every normal subgroup H of G with cyclic quotient G/H, the index $[\mathrm{Ker}\,V_{G\to H}:G_2]$ for the transfer $V_{G\to H}:G\to H/[H,H]$ coincides with the index $[G:H]$;

(B) There exists an automorphism φ of G of order 2 such that $g^{\varphi+1}$ belongs to G_2 for every $g\in G$.

The first condition comes from a property of k on capitulation of ideals: Let K be an unramified abelian p-extension of k and H the corresponding subgroup of G; then $H/[H,H]$ is isomorphic to the p-primary part $\mathrm{Cl}^{(p)}(K)$ of the ideal class group of K by the Artin map for K. We define the capitulation homomorphism $j_{K/k}:C\to\mathrm{Cl}^{(p)}(K)$ by regarding ideals of k naturally as those of K. The Artin maps of k and of K transform this to the homomorphism $\bar{V}_{G\to H}:G/G_2\to H/[H,H]$ which is naturally induced from the transfer $V_{G\to H}$ of G to H (cf. e.g. Miyake [2]); hence the order of $\mathrm{Ker}\,j_{K/k}$ coincides with the index $[\mathrm{Ker}\,V_{G\to H}:G_2]$. The index $[G:H]$ is none other than the degree $[K:k]$. If K/k is a cyclic extension, furthermore, we have

$$|\mathrm{Ker}\,j_{K/k}|=[K:k]\cdot[E_k:\mathrm{N}_{K/k}(E_K)]$$

where E_k and E_K are, respectively, the unit groups of k and of K, and $\mathrm{N}_{K/k}$ is the norm map (cf. e.g. Schmithals [5]). We have $E_k=\{\pm1\}$ because

k is an imaginary quadratic field; (note that the field of the 3rd or the 4th roots of 1 has the class number 1). Hence we have $[E_k : \mathrm{N}_{K/k}(E_K)]=1$ because $[K:k]$ is odd by the assumption. This shows our condition (A). (Cf. [3], Proposition 1.)

Next let us see our group G satisfy the condition (B). Take an element ρ of order 2 in $\mathrm{Gal}(\hat{k}/\boldsymbol{Q})$; it gives the non-trivial automorphism of k. Let us denote the inner automorphism of $\mathrm{Gal}(\hat{k}/\boldsymbol{Q})$ defined by ρ by φ; it induces an automorphism of G and an action of ρ on A. We also have a natural action of ρ on C. The Artin map of C onto A is compatible with these actions of ρ. We have, therefore, the desired result by the next proposition ([3], Proposition 2) due to Suzuki.

Proposition 1. *Let k be a quadratic extension of an algebraic number field k_0 of finite degree, and denote the non-trivial automorphism of k/k_0 by ρ. Let c be an element of the ideal class group* $\mathrm{Cl}(k)$ *of k, and suppose that its order is relatively prime to the class number $|\mathrm{Cl}(k_0)|$ of k_0. Then we have $c^\rho=c^{-1}$.*

3. First we give a rough sketch of the proof of Theorem 1. It is easy to see that there exists an automorphism φ of D of order 2 such that

$$a_i^\varphi=a_i^{-1},\quad c_{i,j}^\varphi=c_{i,j},\quad i=1,\cdots,r,\quad j=i+1,\cdots,r.$$

Let E denote the semi-direct product of D and $\langle\varphi\rangle$; the commutator group $[D,D]$ is normal in E and contained in both of $[E,E]$ and the center of E; hence in particular, E is a non-splitting central extension of $E/[D,D]$. We may, by Proposition 1, identify this quotient group with $\mathrm{Gal}(\tilde{k}/\boldsymbol{Q})$. Put $|[D,D]|=p^n$, and take a series of subgroups of $[D,D]$,

$$U_0=[D,D]\supset U_1\supset U_2\supset\cdots\supset U_n=1,$$

such that $[U_t : U_{t+1}]=p$, $t=0,1,\cdots,n-1$. Then we have a series of non-splitting central extension E/U_{t+1} of E/U_t by a cyclic group U_t/U_{t+1} of order p. We now apply Theorem 1 of Nomura [4] first to the Galois tower $\tilde{k}/k/\boldsymbol{Q}$ to obtain an unramified extension K_1 of k such that it is normal over $\boldsymbol{Q}$ with the Galois group isomorphic to E/U_1; then next do it to $K_1/k/\boldsymbol{Q}$ to obtain K_2, and so on, and finally have an unramified extension $K:=K_n$ of k such that $\mathrm{Gal}(K/\boldsymbol{Q})$ is isomorphic to E. It is clear that $\mathrm{Gal}(K/k)$ is isomorphic to our group D. We have proved our Theorem 1.

The corollary to it is also apparent (cf. e.g. Huppert [1], V, 23.3).

4. Next we study the structure of $G=\mathrm{Gal}(\hat{k}/k)$ where we can see effects of the conditions (A) and (B).

4-1. Let us choose a set of generators of G,

$$G=\langle\alpha_i\,|\,i=1,\cdots,r\rangle,\quad \alpha_i^{\varepsilon(i)}\in G_2=[G,G],\quad i=1,\cdots,r,$$

and put

$$[\alpha_i,\alpha_j]=\gamma_{i,j},\quad 1\leq i<j\leq r,$$

in correspondence to those of $D\cong G/G_3$. We take r subgroups

$$H_i=\langle\alpha_n\,|\,1\leq n\leq r,\ n\neq i\rangle\cdot G_2,\quad i=1,\cdots,r,$$

to utilize the condition (A); apparently G/H_i is cyclic; it is of order $\varepsilon(i)$ and generated by the coset of α_i. For simplicity, we denote the transfer of G to H_i by $V_i := V_{G\to H_i}$, and put $H_\infty := \bigcap_{i=1}^r [H_i, H_i]$. For $x, y \in G$, define

$$\gamma_1(x, y) := [x, y], \quad \gamma_n(x, y) := [\gamma_{n-1}(x, y), y], \quad n=2, 3, 4, \cdots,$$

inductively, and take r subgroups X_i, $i=1, \cdots, r$, of G_3,

$$X_i := \langle \gamma_n(\alpha_j, \alpha_i) \mid 1\le j\le r,\ j\ne i,\ n=2, 3, 4, \cdots\rangle.$$

Lemma 1. (1) *$G_3\cdot[H_i, H_i]=X_i\cdot[H_i, H_i]$ for $i=1, \cdots, r$;*

(2) *If $i\ne j$, then $X_i\subset[H_j, H_j]$ and $X_i\cap X_j\subset H_\infty$;*

(3) *$X_i\cap[H_i, H_i]=X_i\cap H_\infty$ and $X_i\cdot[H_i, H_i]/[H_i, H_i]\cong X_i/X_i\cap H_\infty$ for $i=1, \cdots, r$.*

Proof. Put $W=G/[H_i, H_i]$ for a fixed i. Since H_i contains $G_2=[G, G]$ by definition, every coset $\alpha_n\cdot[H_i, H_i]$ with $n\ne i$ commutes with each of commutators of W. If $m\ne i$ and $n\ne i$, then $[\alpha_m, \alpha_n]\in[H_i, H_i]$. Thus $W_3=[[W, W], W]$ coincides with $X_i\cdot[H_i, H_i]/[H_i, H_i]$. Since $W_3=G_3\cdot[H_i, H_i]/[H_i, H_i]$, we have (1) of the lemma. If $i\ne j$, then we see $\alpha_i\in H_j$ and $[\alpha_m, \alpha_i]\in G_2\subset H_j$ for each m; hence we have $X_i\subset[H_j, H_j]$; we obtain, therefore, $X_i\cap X_j\subset H_\infty$ because $X_j\subset[H_n, H_n]$ for every $n\ne j$. The assertion (2) is proved. (3) is clear because $X_i\subset[H_n, H_n]$ for every $n\ne i$ as we have seen it.

Proposition 2. *Let the notation be as above and denote the natural projection of G onto G/H_∞ by π. Then r subgroups $\pi(X_i)$, $i=1, \cdots, r$, form a direct product in the abelian group $\pi(G_3)=G_3\cdot H_\infty/H_\infty$.*

Proof. We see by (2) of the lemma that the subgroup

$$\langle X_j \mid 1\le j\le r,\ j\ne i\rangle\cdot H_\infty$$

is contained in $[H_i, H_i]$ and hence by (3) that

$$X_i\cap\langle X_j \mid 1\le j\le r,\ j\ne i\rangle\cdot H_\infty\subset H_\infty$$

for each $i=1, \cdots, r$. It is apparent that this implies the proposition.

4-2. For each i, $1\le i\le r$, put

$$M_i := \langle \alpha_1, \cdots, \alpha_i, \alpha_{i+1}^{\varepsilon(i+1)/\varepsilon(i)}, \cdots, \alpha_r^{\varepsilon(r)/\varepsilon(i)}\rangle\cdot G_2.$$

Proposition 3. *Let the notation be as above. Then for each $i=1, \cdots, r$, we have*

(1) *$\mathrm{Im}\, V_i\cap G_2/[H_i, H_i]=V_i(M_i)$ and $\mathrm{Ker}\, V_i\subset M_i$;*

(2) *$[G: M_i]=[\mathrm{Im}\, V_i: V_i(M_i)]=|C^{\varepsilon(i)}|$.*

Proof. For the proof of this proposition, we may assume that $[H_i, H_i]=1$ for simplicity by replacing G, H_i, etc. with their images in the quotient group $G/[H_i, H_i]$. Then H_i is a normal abelian subgroup of G. Put $\alpha:=\alpha_i$. Since G/H_i is a cyclic group and generated by α, we have $V_i(\alpha_i)=\alpha_i^q$, $q:=\varepsilon(i)$, and for $x\in H_i$,

$$V_i(x)=x^{\mathrm{Tr}\langle\alpha\rangle}=x^q\cdot\gamma_1(x, \alpha)^q\cdot\gamma_2(x, \alpha)^{\binom{q}{2}}\cdots\gamma_q(x, \alpha),$$

where $\mathrm{Tr}\,\langle\alpha\rangle=\alpha^{q-1}+\alpha^{q-2}+\cdots+\alpha+1$. (Cf. [3], Lemma 2.) Hence we see $\mathrm{Im}\, V_i\cdot G_2/G_2=\langle\alpha_{i+1}^q, \cdots, \alpha_r^q\rangle\cdot G_2/G_2$, and $|\mathrm{Im}\, V_i\cdot G_2/G_2|=|C^q|$. It is then clear that $\mathrm{Im}\, V_i\cap G_2=V_i(M_i)$. Since we have $[G: M_i]=[\mathrm{Im}\, V_i: V_i(M_i)]$, we conclude $\mathrm{Ker}\, V_i\subset M_i$. The proof is completed.

5. We now see consequences of the condition (B).

Lemma 2. *Under the condition (B), we have, for each $n\ge 1$,*

$$g^\varphi=g^{(-1)^n} \bmod G_{n+1} \quad \text{for } g\in G_n.$$

Hence, in particular, we have

$$G_{2n}=G_{2n}^{1+\varphi}\cdot G_{2n+1} \quad \textit{and} \quad G_{2n+1}=G_{2n+1}^{1-\varphi}\cdot G_{2n+2} \quad \textit{for } n\geq 1.$$

We may easily prove the former half in a straightforward way by mathematical induction on n because it is sufficient to show the case of $g=[h,\alpha]$ with $h\in G_{n-1}$ and $\alpha\in G_1$ for $n\geq 2$ (cf. [3], Lemma 3). The latter half follows from the former because p is odd and we have $x^2=x^{1+\varphi}\cdot x^{1-\varphi}$ for $x\in G_2$.

Proposition 4. *Let the notation be as in* §4, *and suppose that the condition* (B) *is satisfied. Then we have*

$$V_i(M_i)=\operatorname{Im} V_i\cap G_2/[H_i,H_i]\subset X_i^{1-\varphi}\cdot[H_i,H_i]/[H_i,H_i]$$

for $i=1,\cdots,r$ where φ is the automorphism of (B).

Proof. For a finite $\langle\varphi\rangle$-module A of odd order, apparently we have $A=A^{1-\varphi}\cdot A^{1+\varphi}$ and $A^{1-\varphi}\cap A^{1+\varphi}=1$. Hence by (1) of Lemma 1 and (1) of Proposition 3, it is sufficient to show that

$$\operatorname{Im} V_i\cap G_2/[H_i,H_i]\subset G_3^{1-\varphi}\cdot[H_i,H_i]/[H_i,H_i].$$

We have $V_i(g)^\varphi=V_i(g^\varphi)$ for each $g\in G$ by Proposition 4 in §2 of [2]. Hence on the one hand, we have $V_i(g)^\varphi=V_i(g^{-1})=V_i(g)^{-1}$. Suppose that $V_i(g)$ belongs to $G_2/[H_i,H_i]$. Then by the preceding lemma, we have $V_i(g)^\varphi=V_i(g)w$, $w\in G_3\cdot[H_i,H_i]/[H_i,H_i]$, on the other hand. Therefore we see $V_i(g)^2$ belong to $G_3\cdot[H_i,H_i]/[H_i,H_i]$, and hence so does $V_i(g)$ because they are in a p-group for an odd prime p. As we mentioned it at the beginning of the proof, G_3 is decomposed into a direct product of $G_3^{1-\varphi}$ and $G_3^{1+\varphi}$. Since $V_i(g)^\varphi=V_i(g)^{-1}$, we have $V_i(g)\in G_3^{1-\varphi}\cdot[H_i,H_i]/[H_i,H_i]$. The proof is completed.

Theorem 3. *Let k be a quadratic number field and suppose that $r=$ p-rank$(C)\geq 2$, $C=\mathrm{Cl}^{(p)}(k)$. Let the notation be as above and K_i/k the maximal unramified cyclic extension fixed by the subgroup $H_i/[G,G]$ of $\mathrm{Gal}(\tilde{k}/k)$ for $i=1,\cdots,r$. Then we have*

(1) $|\mathrm{Cl}^{(p)}(\tilde{k})|=|C\wedge C|\cdot|G_3|=\{\prod_{i=1}^r \varepsilon(i)^{(r-i)}\}\cdot|G_3|$;

(2) $|G_3|\geq\prod_{i=1}^r[C:C^{[K_i:k]}]/|\operatorname{Ker} j_{K_i/k}|$.

Proof. The first assertion is apparent from Theorem 1 and its corollary. By Proposition 2 we see $|G_3|$ greater than or equal to the product of the orders of $\pi(X_i)$, $i=1,\cdots,r$; each of them is not less than $|V_i(M_i)|$ because of (3) of Lemma 1, (1) of Proposition 3, and Proposition 4. The degree $[K_i:k]$ is equal to $\varepsilon(i)$ by definition. Hence it easily follows from (2) of Proposition 3 that $|V_i(M_i)|$ is equal to the i-th term of the right hand side of (2) of the theorem. The proof is completed.

Proposition 5. *Under the same situation as in Theorem* 3, *we have*

$$p\text{-rank}(\mathrm{Cl}^{(p)}(\tilde{k}))\geq p\text{-rank}(C\wedge C)+p\text{-rank}(G_3^{1-\varphi})$$
$$\geq\binom{r}{2}+\sum_{i=1}^r p\text{-rank}(V_i(M_i)).$$

Proof. By Lemma 2, we easily see $G_2^{1-\varphi}=G_3^{1-\varphi}$ and $G_3^{1+\varphi}=G_4^{1+\varphi}\subset G_2^{1+\varphi}$. Since $G_2^{1+\varphi}\cap G_3^{1-\varphi}=1$, the p-rank of G_2 is the sum of those of $G_2^{1+\varphi}$ and of $G_3^{1-\varphi}$. The p-rank of $G_2^{1+\varphi}$ is not less than p-rank$(C\wedge C)$ because $G_2=G_2^{1+\varphi}\cdot G_3$ by Lemma 2. The first ineqality is proved. It is easy to see that we have $p\text{-rank}(C\wedge C)=\binom{r}{2}$. It is also apparent by Proposition 2 that $G_3^{1-\varphi}\cdot H_\infty/H_\infty$ contains a direct product of $\pi(X_i^{1-\varphi})$, $i=1,\cdots,r$. By (3) of Lemma 1 and Proposition 4, we see that each $\pi(X_i^{1-\varphi})$ contains a subgroup which is isomorphic to $V_i(M_i)$. The latter inequality of Proposition 5 is now also clear.

6. Finally we complete the proof of Theorem 2. Suppose that k is an imaginary quadratic number field. Then $G=\mathrm{Gal}(\hat{k}/k)$ satisfies both of the conditions (A) and (B). Therefore, in particular, we have $|\mathrm{Ker}\, j_{K_i/k}|=[K_i:k]=\varepsilon(i)=p^{e_i}$. It is clear by definition that we have

$$[C:C^{[K_i:k]}]/|\mathrm{Ker}\, j_{K_i/k}|=[C:C^{\varepsilon(i)}]/\varepsilon(i).$$

Let a_i, $i=1,\cdots,r$, be a basis of C such that the exponent of a_i is equal to $\varepsilon(i)$. Then we easily see

$$[C:C^{\varepsilon(i)}]/\varepsilon(i)=|a_i\wedge C|.$$

Since $a_i\wedge C$ is a direct product of $\langle a_n\wedge a_i \mid 1\le n<i\rangle$ and $\langle a_i\wedge a_n \mid i<n\le r\rangle$ for $i=1,\cdots,r$, we have

$$\prod_{i=1}^{r}[C:C^{\varepsilon(i)}]/\varepsilon(i)=|C\wedge C|^2.$$

Hence (1) and (2) of Theorem 2 immediately follow from Theorem 3. The assertion (3) of Theorem 2 follows from (4) of it and Proposition 5 at once. We only need, therefore, to show the final assertion (4). By definition, the quotient M_i/G_2 is of type $(\varepsilon(1),\cdots,\varepsilon(i-1),\varepsilon(i),\cdots,\varepsilon(i))$; here we have $r-i+1$ copies of $\varepsilon(i)$. It follows fromthe condition (A) that the order of the quotient group $\mathrm{Ker}\, V_i/G_2$ is equal to $\varepsilon(i)$. It is apparent, therefore, that the least possible number for $p\text{-rank}(V_i(M_i))$ is equal to

$$r-\max\{n \mid e_1+\cdots+e_n\le e_i\}.$$

Hence by Proposition 5 we obtain the former inequality of our (4). For $i=1$, we have $r-\max\{n \mid e_1+\cdots+e_n\le e_i\}=r-1$. For $i>1$, however, we have $r-\max\{n \mid e_1+\cdots+e_n\le e_i\}\ge r-i+1$. We see, therefore, the latter inequality of (4) of Theorem 2 because $\sum_{i=1}^{r}(r-i+1)=\binom{r+1}{2}$. Theorem 2 is completely proved.

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