12. *Kostant's Theorem for a Certain Class of Generalized Kac-Moody Algebras*

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Introduction. Let $A=(a_{ij})_{i,j\in I}$ be a real $n\times n$ matrix satisfying the following conditions:

(C1) either $a_{ii}=2$ or $a_{ii}\leq 0$;

(C2) $a_{ij}\leq 0$ if $i\neq j$, and $a_{ij}\in \boldsymbol{Z}$ if $a_{ii}=2$;

(C3) $a_{ij}=0$ implies $a_{ji}=0$.

Such a matrix is called a *generalized GCM* (=GGCM). And let $\mathfrak{g}(A)$ be the generalized Kac-Moody algebra (=GKM algebra), over the complex number field $\boldsymbol{C}$, associated to the above GGCM A. Then, we have the root space decomposition: $\mathfrak{g}(A)=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}^{\oplus}\mathfrak{g}_\alpha$, where $\mathfrak{h}$ is the Cartan subalgebra, and Δ the root system of $(\mathfrak{g}(A),\mathfrak{h})$. Let J be a subset of $I^{re}:=\{i\in I \mid a_{ii}=2\}$. And put $\mathfrak{n}_J^{\pm}:=\sum_{\alpha\in\Delta_J^+}^{\oplus}\mathfrak{g}_{\pm\alpha}$, $\mathfrak{u}^{\pm}:=\sum_{\alpha\in\Delta^+(J)}^{\oplus}\mathfrak{g}_{\pm\alpha}$, $\mathfrak{m}:=\mathfrak{n}_J^-\oplus\mathfrak{h}\oplus\mathfrak{n}_J^+$, where $\Delta_J^+:=\Delta\cap\sum_{i\in J}\boldsymbol{Z}_{\geq 0}\alpha_i$, $\Delta^+(J):=\Delta^+\setminus\Delta_J^+$. In this paper, we study the homology $H_j(\mathfrak{u}^-, L(\Lambda))$ of $\mathfrak{u}^-$ and the cohomology $H^j(\mathfrak{u}^+, L(\Lambda))$ of $\mathfrak{u}^+$ with coefficients in the irreducible highest weight $\mathfrak{g}(A)$-module $L(\Lambda)$ with highest weight $\Lambda\in\mathfrak{h}^*$. And we prove "Kostant's homology and cohomology theorem" for symmetrizable GKM algebras associated to GGCMs satisfying the following condition ($\hat{\text{C}}$1) instead of (C1) above:

($\hat{\text{C}}$1) either $a_{ii}=2$ or $a_{ii}=0$.

This result is a generalization of Kostant's Theorem for Kac-Moody algebras, which was proved by J. Lepowsky and H. Garland ([2] and [5]), or the classical result of B. Kostant himself [4] for finite dimensional complex semi-simple Lie algebras.

§ 1. Preliminaries for GKM algebras. We prepare some basic results for GKM algebras which will be needed later. For details, see [1] and [3]. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM A, with the Cartan subalgebra $\mathfrak{h}$, simple roots $\Pi=\{\alpha_i\}_{i\in I}$, and simple co-roots $\Pi^\vee=\{\alpha_i^\vee\}_{i\in I}$. From now on, we always assume that the GGCM $A=(a_{ij})_{i,j\in I}$ is symmetrizable, and that J is a subset of $I^{re}=\{i\in I \mid a_{ii}=2\}$. We call an $\mathfrak{h}$-module V *$\mathfrak{h}$-diagonalizable* if V admits a weight space decomposition: $V=\sum_{\lambda\in\mathcal{P}(V)}^{\oplus}V_\lambda$, where $\mathcal{P}(V)$ is the set of all weights of V.

Definition ([6]). $\mathcal{O}_J$ is the category of all $\mathfrak{m}$-modules whose objects V satisfy the following:

(1) V is $\mathfrak{h}$-diagonalizable;

(2) the weight space V_μ is finite dimensional for all $\mu\in\mathcal{P}(V)$;

(3) there exist a finite number of elements λ_i $(1\leq i\leq s)$ in $\mathfrak{h}^*:=$

$\mathrm{Hom}_{C}(\mathfrak{h}, \boldsymbol{C})$ such that $\mathcal{P}(V) \subset \bigcup_{i=1}^{s} D(\lambda_i)$, where $D(\lambda_i) := \{\lambda_i - \beta \mid \beta \in Q_+ = \sum_{j \in I} \boldsymbol{Z}_{\geq 0} \alpha_j\}$ $(1 \leq i \leq s)$.

(4) Viewed as an $\mathfrak{m}$-module, V is a direct sum of irreducible highest weight $\mathfrak{m}$-modules $L_{\mathfrak{m}}(\lambda)$ with highest weight $\lambda \in P_J^+ := \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle \in \boldsymbol{Z}_{\geq 0}$ $(i \in J)\}$.

Note that the category $\mathcal{O}_J$ is closed under the operations of taking submodules, quotients, and finite direct sums. Moreover, a tensor product of a finite number of modules from $\mathcal{O}_J$ is again in the category $\mathcal{O}_J$, due to [3, Theorem 10.7 b)].

The following proposition plays a fundamental role in this paper.

Proposition I ([6]). *Let $\Lambda \in P^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0$ $(i \in I)$, and $\langle \lambda, \alpha_i^\vee \rangle \in \boldsymbol{Z}_{\geq 0}$ if $a_{ii} = 2\}$. Then, $L(\Lambda)$ and $(\bigwedge^j \mathfrak{u}^-) \otimes_{C} L(\Lambda)$ $(j \geq 0)$ are in the category $\mathcal{O}_J$, where $\bigwedge^j \mathfrak{u}^-$ is the exterior algebra of degree j over $\mathfrak{u}^-$ $(j \geq 0)$, and is an $\mathfrak{m}$-module by the adjoint action since $[\mathfrak{m}, \mathfrak{u}^-] \subset \mathfrak{u}^-$.*

Now, we introduce the algebra $\mathcal{E}_J$ of "formal $\mathfrak{m}$-characters" of $\mathfrak{m}$-modules from the category $\mathcal{O}_J$. The elements of $\mathcal{E}_J$ are series of the form $\sum_{\lambda \in P_J^+} c_\lambda e_{\mathfrak{m}}(\lambda)$, where $c_\lambda \in \boldsymbol{C}$ and $c_\lambda = 0$ for λ outside a finite union of the sets of the form $D(\mu)$ $(\mu \in \mathfrak{h}^*)$. Here, the elements $e_{\mathfrak{m}}(\lambda)$ are called *formal* $\mathfrak{m}$-*exponentials*. They are linearly independent and are in one-one correspondence with the elements $\lambda \in P_J^+$.

For a module V in the category $\mathcal{O}_J$, we define the *formal* $\mathfrak{m}$-*character* $\mathrm{ch}_{\mathfrak{m}} V$ of V by $\mathrm{ch}_{\mathfrak{m}} V := \sum_{\lambda \in P_J^+} [V : L_{\mathfrak{m}}(\lambda)] e_{\mathfrak{m}}(\lambda)$, where $[V : L_{\mathfrak{m}}(\lambda)]$ is the "multiplicity" of $L_{\mathfrak{m}}(\lambda)$ in V (see [3, Ch. 9, Lemma 9.6]). Note that, for a module V in the category $\mathcal{O}_J$, $[V : L_{\mathfrak{m}}(\lambda)]$ $(\lambda \in P_J^+)$ is finite and so $\mathrm{ch}_{\mathfrak{m}} V$ is an element of the algebra $\mathcal{E}_J$. Then, the multiplication of $\mathcal{E}_J$ is defined by $e_{\mathfrak{m}}(\lambda) \cdot e_{\mathfrak{m}}(\mu) := \mathrm{ch}_{\mathfrak{m}}(L_{\mathfrak{m}}(\lambda) \otimes_{C} L_{\mathfrak{m}}(\mu))$ $(\lambda, \mu \in P_J^+)$. Thus, $\mathcal{E}_J$ becomes a commutative associative algebra over $\boldsymbol{C}$.

Especially when $J = \phi$, the algebra $\mathcal{E}_J$ is nothing but the algebra $\mathcal{E}$ in [3, Ch. 9], since in this case $\mathfrak{m} = \mathfrak{h}$, $P_J^+ = \mathfrak{h}^*$, and $e_{\mathfrak{m}}(\lambda) = e(\lambda)$ $(\lambda \in P_J^+ = \mathfrak{h}^*)$. Now, let $(\cdot \mid \cdot)$ be a fixed *standard bilinear form* on $\mathfrak{h}^*$, Π^{im} (resp. Π^{re}) be the subset $\{\alpha_i \in \Pi \mid a_{ii} \leq 0$ (resp. $a_{ii} = 2)\}$ of Π, and $W \subset GL(\mathfrak{h}^*)$ be the *Weyl group* generated by the fundamental reflections r_i defined by $\alpha_i \in \Pi^{re}$. And let $\mathfrak{S}$ be the set of all sums of distinct pairwise perpendicular elements, with respect to $(\cdot \mid \cdot)$, from Π^{im}. Note that $\{0\} \cup \Pi^{im}$ is contained in $\mathfrak{S}$. Then, we know the following character formula.

Theorem I ([1] and [3]). *Let $\Lambda \in P^+$ and $\mathfrak{S}(\Lambda) := \{\lambda \in \mathfrak{S} \mid (\lambda \mid \Lambda) = 0\}$. And we put*

$$S_\Lambda := e(\Lambda + \rho) \cdot \sum_{\beta \in \mathfrak{S}(\Lambda)} \varepsilon(\beta) e(-\beta), \qquad R := \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\mathrm{mult}(\alpha)},$$

where $\varepsilon(\beta) = (-1)^m$ if $\beta \in \mathfrak{S}$ is a sum of m elements from Π^{im}, $\rho \in \mathfrak{h}^*$ is a fixed element such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ $(i \in I)$, and $\mathrm{mult}(\alpha) := \dim_{C} \mathfrak{g}_\alpha$ $(\alpha \in \Delta^+)$. *Then, there holds in the algebra $\mathcal{E} = \mathcal{E}_\phi$,*

$$e(\rho) \cdot R \cdot \mathrm{ch}\, L(\Lambda) = \sum_{w \in W} (\det w) w(S_\Lambda),$$

with $w(e(\mu)) := e(w(\mu))$ $(\mu \in \mathfrak{h}^*)$.

Corollary I ([1] and [3]). *We put $S := e(\rho) \cdot \sum_{\beta \in \mathfrak{S}} \varepsilon(\beta) e(-\beta)$. Then,*

$$e(\rho)\cdot R=\sum_{w\in W}(\det w)w(S).$$

Remark 1.1. The above statement of Theorem I (resp. Corollary I) is the corrected version of Theorem 11.13.3 (resp. Corollary 11.13.2) in [3].

§ 2. Homology and cohomology of GKM algebras. In this section, we will review the notion of homology and cohomology of Lie algebras. Let $L(\Lambda)$ be the irreducible highest weight $\mathfrak{g}(A)$-module with highest weight $\Lambda\in P^+$. Then, the vector space $C^j(\mathfrak{u}^+, L(\Lambda))$ of j cochains is defined by $C^j(\mathfrak{u}^+, L(\Lambda)):=\mathrm{Hom}_C^c(\wedge^j\mathfrak{u}^+, L(\Lambda))$, and is an $\mathfrak{m}$-module in a usual sense $(j\geq 0)$. Here, for $\mathfrak{h}$-diagonalizable modules $V=\sum_{\lambda\in\mathfrak{h}^*}^{\oplus}V_\lambda$ and $W=\sum_{\mu\in\mathfrak{h}^*}^{\oplus}W_\mu$ with finite dimensional weight spaces, we put

$$\mathrm{Hom}_C^c(V, W):=\{f\in \mathrm{Hom}_C(V, W)\,|\,f(V_\lambda)=0 \text{ for all but finitely many weights } \lambda\in\mathfrak{h}^* \text{ of } V\}.$$

The coboundary operator $d^j: C^j(\mathfrak{u}^+, L(\Lambda))\to C^{j+1}(\mathfrak{u}^+, L(\Lambda))$ is defined by

$$(d^jf)(x_1\wedge\cdots\wedge x_j\wedge x_{j+1}):=\sum_{i=1}^{j+1}(-1)^i x_i(f(x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_{j+1}))$$
$$+\sum_{1\leq r<t\leq j+1}(-1)^{r+t}f([x_r, x_t]\wedge x_1\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_t\wedge\cdots\wedge x_{j+1}),$$

where $x_1,\cdots,x_{j+1}\in\mathfrak{u}^+$, $f\in C^j(\mathfrak{u}^+, L(\Lambda))$, and the symbol $\hat{x}_i$ indicates a term to be omitted. The cohomology of this complex $\{C^j(\mathfrak{u}^+, L(\Lambda)), d^j\}_{j\in \boldsymbol{Z}}$ is called the j^{th} cohomology of $\mathfrak{u}^+$ with coefficients in $L(\Lambda)$, and is denoted by $H^j(\mathfrak{u}^+, L(\Lambda))$. Then, $H^j(\mathfrak{u}^+, L(\Lambda))$ is also an $\mathfrak{m}$-module, since the coboundary operator d^j commutes with the action of $\mathfrak{m}$.

For the homology, we define the vector space $C_j(\mathfrak{u}^-, L(\Lambda))$ of j chains to be $\wedge^j\mathfrak{u}^-\otimes_C L(\Lambda)$, which is an $\mathfrak{m}$-module in a usual sense $(j\geq 0)$. The boundary operator $d_j: C_j(\mathfrak{u}^-, L(\Lambda))\to C_{j-1}(\mathfrak{u}^-, L(\Lambda))$ is defined by

$$d_j(y_1\wedge\cdots\wedge y_j\otimes v):=\sum_{i=1}^{j}(-1)^i(y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_j)\otimes y_i(v)$$
$$+\sum_{1\leq r<t\leq j}(-1)^{r+t}([y_r, y_t]\wedge y_1\wedge\cdots\wedge\hat{y}_r\wedge\cdots\wedge\hat{y}_t\wedge\cdots\wedge y_j)\otimes v,$$

where $y_1,\cdots,y_j\in\mathfrak{u}^-$, $v\in L(\Lambda)$. And we have the similar situation as in the case of cohomology.

Remark 2.1. In this paper, the cohomology $H^j(\mathfrak{u}^+, L(\Lambda))$ of $\mathfrak{u}^+$ is different from the usual one, since we have used $\mathrm{Hom}_C^c(\wedge^j\mathfrak{u}^+, L(\Lambda))$ instead of $\mathrm{Hom}_C(\wedge^j\mathfrak{u}^+, L(\Lambda))$ as $C^j(\mathfrak{u}^+, L(\Lambda))$ $(j\geq 0)$.

§ 3. Kostant's Theorem for GKM algebras. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a symmetrizable GGCM $A=(a_{ij})_{i,j\in I}$. For a subset J of I^{re}, we put $A_J:=(a_{ij})_{i,j\in J}$, which is a generalized Cartan matrix (=GCM). Then, since the triple $(\mathfrak{h}, \{\alpha_i\}_{i\in J}, \{\alpha_i^\vee\}_{i\in J})$ is a *realization* (but not a minimal one) of the GCM A_J, the subalgebra $\mathfrak{m}$ of $\mathfrak{g}(A)$ can be regarded as a Kac-Moody algebra associated to the GCM A_J, whose Cartan subalgebra is $\mathfrak{h}$. So, the well-known representation theory for Kac-Moody algebras is also applicable to the subalgebra $\mathfrak{m}$ of $\mathfrak{g}(A)$ (cf. [3, Chs. 9 and 10]).

3.1. Results of L. Liu. Here, we rewrite, in the case of GKM algebras, some of Liu's results on $\mathfrak{m}$-modules $H_j(\mathfrak{u}^-, L(\Lambda))$ and $H^j(\mathfrak{u}^+, L(\Lambda))$ for Kac-Moody algebras. The proofs of these results for GKM algebras need no modifications. For details, see [6] and also the appendix of [2].

Proposition 3.1 ([6]). *$H^j(\mathfrak{u}^+, L(\Lambda))$ is isomorphic to $H_j(\mathfrak{u}^-, L(\Lambda))$ as $\mathfrak{m}$-modules for any $\Lambda\in P^+$ and $j\in \boldsymbol{Z}_{\geq 0}$.*

Due to this, it is enough for us to consider $H_j(\mathfrak{u}^-, L(\Lambda))$ ($j \geq 0$) only. And, since $L(\Lambda)$ and $(\wedge^j \mathfrak{u}^-)\otimes_C L(\Lambda)$ are in the category $\mathcal{O}_J$ by Proposition I, $H_j(\mathfrak{u}^-, L(\Lambda))$ is also in $\mathcal{O}_J$, and is a direct sum of $L_{\mathfrak{m}}(\mu)$, $\mu \in P_J^+$, as $\mathfrak{m}$-modules ($j \geq 0$). Furthermore, we have

Proposition 3.2 ([6]). *Let* $(\cdot|\cdot)$ *be a standard bilinear form on* $\mathfrak{h}^*$. *Then, for any* $\Lambda \in P^+$ *and* $j \in Z_{\geq 0}$, *every* $\mathfrak{m}$*-irreducible component of* $H_j(\mathfrak{u}^-, L(\Lambda))$ *is of the form* $L_{\mathfrak{m}}(\mu)$, $\mu \in P_J^+$, *with* $(\mu+\rho|\mu+\rho)=(\Lambda+\rho|\Lambda+\rho)$.

3.2. Main theorem. From now on, we assume that the symmetrizable GGCM $A=(a_{ij})_{i,j\in I}$ satisfies the following condition $(\hat{\mathrm{C}}1)$:

$(\hat{\mathrm{C}}1)$ either $a_{ii}=2$ or $a_{ii}=0$ $(i \in I)$.

Then, from Theorem I and Corollary I, we get the following.

Lemma 3.1. $e(\rho)\cdot \mathrm{ch}(\wedge \mathfrak{n}^-)=\mathrm{ch}(\sum^{\oplus}_{\beta\in\mathfrak{S}} L(\rho-\beta))$, *with* $\mathfrak{n}^- := \sum^{\oplus}_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}$.

Remark 3.1. By the condition $(\hat{\mathrm{C}}1)$, $\rho-\beta\in P^+$ for all $\beta \in \mathfrak{S}$.

From the above lemma, it follows that, for every $\Lambda \in \mathfrak{h}^*$,

$$e(\rho)\cdot \mathrm{ch}((\wedge \mathfrak{n}^-)\otimes_C L(\Lambda))=\mathrm{ch}((\textstyle\sum^{\oplus}_{\beta\in\mathfrak{S}} L(\rho-\beta))\otimes_C L(\Lambda)).$$

Therefore, μ is a weight of $(\wedge \mathfrak{n}^-)\otimes_C L(\Lambda)$ if and only if $\mu+\rho$ is a weight of $(\sum^{\oplus}_{\beta\in\mathfrak{S}} L(\rho-\beta))\otimes_C L(\Lambda)$, and moreover, they have the same multiplicity. Using this fact, we can show

Lemma 3.2. *Let* $\Lambda \in P^+$. *Assume that* μ *is a weight of* $(\wedge^j \mathfrak{u}^-)\otimes_C L(\Lambda)$ *for some* $j \geq 0$, *and satisfies* $(\mu+\rho|\mu+\rho)=(\Lambda+\rho|\Lambda+\rho)$. *Then,*

(a) *there exist a* $\beta\in\mathfrak{S}(\Lambda)$ *and a* $w\in W(J) := \{w \in W | w(\Delta^-)\cap\Delta^+ \subset \Delta^+(J)\}$, *such that* $\ell(w)+ht(\beta)=j$ *and* $\mu=w(\Lambda+\rho-\beta)-\rho$;

(b) *the multiplicity of* μ *in* $(\wedge^j \mathfrak{u}^-)\otimes_C L(\Lambda)$ *is equal to one.*

Here, $\ell(w)$ *is the length of* $w \in W$, *and* $ht(\beta)=m$ *if* $\beta\in\mathfrak{S}$ *is a sum of* m *distinct elements from* Π^{im}.

By Proposition 3.2 and Lemma 3.2, we have the following.

Proposition 3.3. *Let* $\Lambda \in P^+$ *and* $j \in Z_{\geq 0}$. *If* $L_{\mathfrak{m}}(\mu)$ $(\mu \in P_J^+)$ *is an* $\mathfrak{m}$*-irreducible component of* $H_j(\mathfrak{u}^-, L(\Lambda))$, *then*

(a) $\mu=w(\Lambda+\rho-\beta)-\rho$, *for some* $\beta\in\mathfrak{S}(\Lambda)$ *and some* $w\in W(J)$, *such that* $\ell(w)+ht(\beta)=j$;

(b) $L_{\mathfrak{m}}(\mu)$ *occurs with multiplicity one as* $\mathfrak{m}$*-irreducible components of* $H_j(\mathfrak{u}^-, L(\Lambda))$.

Now, from Theorem I and the Euler-Poincaré principle (cf. [2]), we get the following.

Lemma 3.3. *For* $\Lambda \in P^+$, *there holds in the algebra* $\mathcal{E}_J$,

$$\textstyle\sum_{j\geq 0}(-1)^j \mathrm{ch}_{\mathfrak{m}}(H_j(\mathfrak{u}^-, L(\Lambda)))=\sum_{\beta\in\mathfrak{S}(\Lambda)}\varepsilon(\beta)\sum_{w\in W(J)}(\det w)e_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho).$$

Remark 3.2. For $w\in W(J)$ and $\beta\in\mathfrak{S}$, $w(\Lambda+\rho-\beta)-\rho\in P_J^+$.

By Proposition 3.3 and Lemma 3.3, we have

Proposition 3.4. *Let* $\Lambda \in P^+$ *and fix* $j \in Z_{\geq 0}$. *For each* $\beta\in\mathfrak{S}(\Lambda)$ *and* $w\in W(J)$ *such that* $\ell(w)+ht(\beta)=j$, *we put* $\mu:=w(\Lambda+\rho-\beta)-\rho$. *Then,* $L_{\mathfrak{m}}(\mu)$ *occurs as* $\mathfrak{m}$*-irreducible components of* $H_j(\mathfrak{u}^-, L(\Lambda))$.

Summarizing Propositions 3.1, 3.3, and 3.4, we obtain our main theorem.

Theorem 3.1. *Let* $\mathfrak{g}(A)$ *be the* GKM *algebra associated to a symmetrizable* GGCM $A=(a_{ij})_{i,j\in I}$ *satisfying* $(\hat{\mathrm{C}}1)$. *And let* $L(\Lambda)$ *be the irreducible highest weight* $\mathfrak{g}(A)$*-module with highest weight* $\Lambda\in P^+$. *We assume that the subset* J *of* I *is contained in* I^{re}. *Then, for* $j\geq 0$,

$$H^j(\mathfrak{u}^+, L(\Lambda))\cong H_j(\mathfrak{u}^-, L(\Lambda))\cong \sum\nolimits^{\oplus}_{\beta\in\mathfrak{S}(\Lambda)} \sum\nolimits_{\substack{w\in W(J)\\ \ell(w)=j-ht(\beta)}} L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho)$$

as $\mathfrak{m}$*-modules. Here,* $L_{\mathfrak{m}}(\mu)$ $(\mu\in P_J^+)$ *is the irreducible highest weight* $\mathfrak{m}$*-module with highest weight* μ.

Remark 3.3. When A is a GCM (i.e., $a_{ii}=2$ for all i), $\mathfrak{S}(\Lambda)$ consists of only one element $0\in\mathfrak{h}^*$. Hence, in this case, the above theorem is nothing but the well-known Kostant's Theorem for Kac-Moody algebras (see [2] and [5]).

References

[1] R. Borcherds: Generalized Kac-Moody algebras. J. Algebra, **115**, 501–512 (1988).

[2] H. Garland and J. Lepowsky: Lie algebra homology and the Macdonald-Kac formulas. Invent. math., **34**, 37–76 (1976).

[3] V. G. Kac: Infinite Dimensional Lie Algebras. 3rd ed., Cambridge University Press, Cambridge (1990).

[4] B. Kostant: Lie algebra cohomology and generalized Borel-Weil theorem. Annals of Math., **74**, 329–387 (1961).

[5] J. Lepowsky: Generalized Verma modules, loop space cohomology and Macdonald-type identities. Ann. scient. Éc. Norm. Sup., **12**, 169–234 (1979).

[6] L. Liu: Kostant's formula for Kac-Moody Lie algebras. Yale University dissertation (1990).