75. On the Dirichlet Form on a Lusinian State Space

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1. Introduction. The Dirichlet forms on locally compact state spaces have been studied by many authors. Recently this theory of Dirichlet forms has been extended to non-locally compact state spaces. Albeverio and Ma [1] gave a necessary and sufficient condition for the Dirichlet form on a metrizable topological state space to be associated with a special standard process. They called this Dirichlet form quasi-regular (cf. [3]). On the other hand, Shigekawa and Taniguchi [12] showed that various results known for locally compact state spaces, such as the Beurling-Deny formula, the uniqueness of the α-potentials, are also valid for Lusinian separable metric state spaces. The key lemma in [12] is a uniqueness statement for a measure which charges no set of zero capacity. Its proof needs the Gel'fand compactification (cf. [4], [9]). To use the Gel'fand compactification we must assume that there exists a dense subset consisting of continuous functions in the domain of the Dirichlet form. However, this assumption is not necessary for the existence of the associated process (cf. [1]). In fact Albeverio, Röckner and Ma [3] showed the same results for quasi-Dirichlet form on general state spaces. They also used another type of compactification (cf. [10]).

In this note we shall show for the quasi-regular Dirichlet form the uniqueness statement of a measure charging no set of zero capacity without using any type of compactification.

2. Preliminary. Let X be a Lusinian separable metric space and let $\mathcal{B}(X)$ be its topological Borel field. Let ρ be its metric. We fix a probability measure m on $(X, \mathcal{B}(X))$ such that $\mathrm{supp}[m] = X$.

We consider a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ (for its definition see e.g. [8]). We set

(2.1) $$\mathcal{E}_1(f, g) \equiv \mathcal{E}(f, g) + (f, g), \quad f, g \in \mathcal{F},$$

where $(\cdot, \cdot)$ denotes the inner product of $L^2(X, m)$.

For an open subset G of X and any subset A of X, we define

(2.2) $$\mathrm{Cap}(G) \equiv \inf\{\mathcal{E}_1(u, u) ; u \in \mathcal{F} \text{ and } u \geq 1 \ m\text{-a.e. on } G\},$$

(2.2) $$\mathrm{Cap}(A) \equiv \inf\{\mathrm{Cap}(G) ; G \text{ is open and } A \subset G\}.$$

Then we can show that this Cap is a Choquet capacity.

A statement depending on $x \in A$ is said to hold "*quasi-everywhere*" or simply "*q.e.*", if it holds on A except for a set of zero capacity with respect to Cap. A function $u : X \to \boldsymbol{R}$ is said to be quasi-continuous if there exists a decreasing sequence $\{G_n\}_{n=1}^{\infty}$ of open sets such that $\mathrm{Cap}(G_n) \downarrow 0$, and $u|_{X \setminus G_n}$ is continuous on each $X \setminus G_n$.

3. The main theorem. We assume that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the following conditions:

(A.1) Cap($\cdot$) is tight; for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $\mathrm{Cap}(X \setminus K) < \varepsilon$.

(A.2) There exists an $\mathcal{E}_1$-dense subset $\mathcal{F}_0$ of $\mathcal{F}$ consisting of quasi-continuous functions.

(A.3) There exists a countable subset $\mathcal{B}_0$ of $\mathcal{F}_0$ and a subset $N \in \mathcal{B}(X)$ with $\mathrm{Cap}(N) = 0$ such that

$$\sigma\{u \in \mathcal{B}_0\} \supset \mathcal{B}(X) \cap (X \setminus N).$$

These conditions (A.1-3) are introduced by Albeverio and Ma [1]. Shigekawa and Taniguchi [12] used instead of (A.2) the following condition:

(A.2') There exists an $\mathcal{E}_1$-dense subset $\mathcal{F}_0$ of $\mathcal{F}$ consisting of bounded continuous functions.

However, the condition (A.1,2',3) are not necessary for the existence of the associated process. In fact, Albeverio and Ma proved that the above conditions (A.1-3) are necessary and sufficient for $(\mathcal{E}, \mathcal{F})$ to be associated with a special standard process ([1], [7]). They called this Dirichlet form quasi-regular (cf. [3]). It is further known that, if a cemetery point Δ is adjoined to X as an isolated point in $X_\Delta = X \cup \Delta$, this process is a Hunt process (cf. [1], [2], [12]).

In this note, we assume in addition to (A.1-3) that

(A.4) $\mathcal{F}_0$ contains $u = 1$ *q.e.*,

and show the uniqueness of a measure charging no set of zero capacity, improving the result of Shigakawa and Taniguchi with (A.1,2',3,4). Without loss of generality, we may assume that $\mathcal{F}_0$ is a $\boldsymbol{Q}$-algebra and closed under the operations $\vee 0$ and $\wedge 1$.

Theorem 1. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ and assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (A.1-4). *Let K be a compact subset of X. Denote the subset of $\mathcal{F}_0$ of the bounded functions by $b\mathcal{F}_0$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $b\mathcal{F}_0$ with $0 \leq f_n \leq 1$ such that*

$$f_n \longrightarrow I_K, \quad q.e.$$

In particular, if μ and ν are finite measures on $(X, \mathcal{B}(X))$ charging no set of zero capacity such that

$$\int_X f d\mu = \int_X f d\nu, \quad f \in b\mathcal{F}_0,$$

then $\mu = \nu$.

Lemma 1. *Let X be a separable metric space. Then $\mathcal{F}$ is separable with respect to the $\mathcal{E}_1$-norm.*

Proof. See [8, Section 1.3].

Lemma 2. *Let F be a set. Consider a countable subset G of F and a countable collection S of mappings s of $F \times F$ to F. Then there exists a countable set H such that*

(a) $G \subset H \subset F$,

(b) $s(H \times H) \subset H, \quad s \in S.$

Proof. See [8, Lemma 6.1.1].

Lemma 3. *There exists a countable subset $\mathcal{H}$ of $\mathcal{F}_0$ consisting of* *quasi-continuous fnctions such that*

(1) $\mathscr{H}$ *contains* $u = 1$ *q.e.,*

(2) $\mathscr{B}_0 \subset \mathscr{H}$,

(3) $\mathscr{H}$ *is dense in* $\mathscr{F}$ *with respect to the* $\mathscr{E}_1$*-norm,*

(4) $\mathscr{H}$ *is an algebra over* $\boldsymbol{Q}$,

(5) $\mathscr{H}$ *is closed under the operations* $\vee 1$ *and* $\wedge 0$.

Proof. We define the mappings from $\mathscr{F}_0 \times \mathscr{F}_0$ into $\mathscr{F}_0$ as follows: $s_1(f, g) = f + g$, $s_2(f, g) = fg$, $s_3(f, g) = f \vee 0$, $s_4(f, g) = f \wedge 1$, $s^a(f, g) = af$, $a \in \boldsymbol{Q}$. We set $\mathscr{S} = \{s_1, s_2, s_3, s_4, s^a ; a \in \boldsymbol{Q}\}$. By Lemma 1 and (A.2), we can choose a countable subset $\{u_n\}_{n=1}^{\infty}$ of $\mathscr{F}_0$ such that $\{u_n\}_{n=1}^{\infty}$ is dense in $\mathscr{F}$ with respect to the $\mathscr{E}_1$-norm, and we suppose $\{u_n\}_{n=1}^{\infty}$ contains a function $u \in \mathscr{F}_0$ which is equal to 1 quasi-everywhere. We can apply Lemma 2 with $S = \mathscr{S}$, $F = \mathscr{F}_0$ and $G = \{u_n\}_{n=1}^{\infty} \cup \mathscr{B}_0$ to get $\mathscr{H}$ (e.g. [8, Lemma 6.1.2]).

In the following, $\mathscr{H}$ denotes a subset of $\mathscr{F}_0$ which has the properties in Lemma 3.

Lemma 4. *$\mathscr{H}$ separates the points of $X \setminus N$.*

Proof. Suppose that there exist $x, y \in X \setminus N$ such that $f(x) = f(y)$ for all $f \in \mathscr{H}$. We must have $x, y \in \cap_{f \in \mathscr{H}} f^{-1}(f(x))$. Since $\mathscr{H}$ includes $\mathscr{B}_0$ and $\mathscr{B}_0$ generates the Borel sets of $X \setminus N$, $\cap_{f \in \mathscr{H}} f^{-1}(f(x))$ is an atom of $X \setminus N$. Hence $x, y \in \{x\}$. This means $x = y$ (cf. [1. Lemma A.7]).

Lemma 5. *There exists a sequence of closed subsets $\{F_k^{(1)}\}_k$ of X such that*

$$\mathscr{H} \subset C(\{F_k^{(1)}\}_k),$$

and

$$\mathrm{Cap}(X \setminus F_k^{(1)}) \to 0, \quad as \quad k \to \infty,$$

where

$$C(\{F_k^{(1)}\}_k) \equiv \{u ; u|_{F_k^{(1)}} \text{ is continuous for each } k\}.$$

Proof. See [8, Theorem 3.1.2].

Proof of Theorem 1. By the condition (A.3), $\mathrm{Cap}(N) = 0$. So there is an increasing sequence of closed subsets $\{F_k^{(2)}\}_k$ of X such that

$$N \subset \bigcap_{k=1}^{\infty} (X \setminus F_k^{(2)}),$$

and

$$\mathrm{Cap}(X \setminus F_k^{(2)}) \to 0, \quad \text{as} \quad k \to \infty.$$

By the condition (A.1), there is an increasing sequence of compact subsets $F_k^{(3)}$ of X such that

$$\mathrm{Cap}(X \setminus F_k^{(3)}) \to 0, \quad \text{as} \quad k \to \infty.$$

By Lemma 5, there is an increasing sequence of closed subsets $F_k^{(1)}$ of X such that

$$\mathscr{H} \subset C(\{F_k^{(1)}\}_k),$$
$$\mathrm{Cap}(X \setminus F_k^{(1)}) \to 0, \quad \text{as} \quad k \to \infty.$$

Now we set

$$F_k \equiv F_k^{(1)} \cap F_k^{(2)} \cap F_k^{(3)}.$$

Then $\{F_k\}_{k=1}^{\infty}$ is an increasing sequence of compact sets such that

$$\mathrm{Cap}(X \setminus F_k) \to 0, \quad \text{as} \quad k \to \infty.$$

We have this by subadditivity of the Choquet capacity

$$\mathrm{Cap}(X\setminus F_k) \le \mathrm{Cap}(X\setminus F_k^{(1)}) + \mathrm{Cap}(X\setminus F_k^{(2)}) + \mathrm{Cap}(X\setminus F_k^{(3)}) \to 0, \quad \text{as} \quad k\to\infty.$$

For each compact subset $K \subset X$, we set

$$K_k \equiv K \cap F_k,$$

and

$$G_k^l \equiv \left\{x \in X \mid \rho(x, K_k) < \frac{1}{l}\right\}, \quad l \in \boldsymbol{N}.$$

For a fixed $k \in \boldsymbol{N}$, $G_k^l \cap F_k$ is an open set and K_k is a closed set with respect to the relative topology in F_k. By Urysohn's lemma, there is a continuous function g_k^l defined on F_k such that

$$0 \le g_k^l \le 1, \quad \text{on } F_k,$$
$$g_k^l(x) = 1, \quad x \in K_k,$$
$$g_k^l(x) = 0, \quad x \in F_k \setminus G_k^l.$$

Since $\mathscr{H}$ is a $\boldsymbol{Q}$-algebra and separates the points of $X\setminus N$ by Lemma 4, $\mathscr{H}|_{F_k}$ is also a $\boldsymbol{Q}$-algebra and separates the points of F_k. Therefore by the Stone-Weierstrass theorem, $\mathscr{H}|_{F_k}$ is dense in $C(F_k)$ with respect to the uniform norm. We can choose $h_k^l \in \mathscr{H}$ such that

$$\| g_k^l - h_k^l|_{F_k} \|_\infty < \frac{1}{4}.$$

We set

$$f_k^l \equiv 0 \vee \left(\left(2h_k^l - \frac{1}{2}\right) \wedge 1\right).$$

Then f_k^l is contained in $\mathscr{H}$, and has the following properties:

$$0 \le f_k^l \le 1, \quad \text{on } X,$$
$$f_k^l(x) = 1, \quad x \in K_k,$$
$$f_k^l(x) = 0, \quad x \in F_k \setminus G_k^l.$$

We consider the sequence $\{f_k^l\}_{k,l}$. If x is contained in $K \cap (\cup_{k=1}^\infty F_k)$, then there is a number $k_0 \in \boldsymbol{N}$ such that the K_k contain x for all $k_0 > L$. Therefore, for all $k > k_0$ and all l, $f_k^l(x) = 1$. On the other hand, if x is contained in $K^c \cap (\cup_{k=1}^\infty F_k)$, then we can choose $N \in \boldsymbol{N}$ such that

$$x \notin \bigcup_{k=1}^{\infty} G_k^l, \quad \text{for } l > N,$$
$$x \in F_k, \quad \text{for } k > N.$$

Therefore, for all k, $l > N$, $f_k^l(x) = 0$. Thus, if $x \in \cup_{k=1}^\infty F_k$, then

$$f_k^l(x) \to I_K, \quad \text{as } k,\, l \to \infty.$$

Since $\mathrm{Cap}(X\setminus \cup_{k=1}^\infty F_k) = 0$, we have

$$f_k^l \to I_K, \quad q.e. \text{ in } X.$$

The last assertion of Theorem 1 follows from the fact that the measures on a Lusinian space are characterized by compact sets [6, III, Theorem 38].

Remark. By the same method of Theorem 1, we can also show the following statement (cf. [12] Lemma 1.3).

Let E_i, $i = 1, 2$ be disjoint closed sets in X. Then there is a sequence of functions $\{u_n\}_{n=1}^\infty \subset \mathscr{F}_{cpt}$ such that $0 \le u_n \le u_{n+1} \le 1$,

$$u_n = 0 \quad \textit{q.e. on } E_1 \quad \textit{and} \quad u_n \to 1 \textit{ q.e. } \quad \textit{on } E_2,$$

where $\mathscr{F}_{cpt}$ *is a family of the function* $\mathscr{F}$ *with compact support.*

In fact, we can construct a sequence of functions $\{f_k\}_{k=1}^{\infty} \subset \mathscr{H}$ having the following properties, $0 \leq f_k \leq 1$ on X, $f_k(x) = 1$ for $x \in F_k \cap E_1$, $f_k(x) = 0$ for $x \in F_k \cap E_2$, where $\{F_k\}_{k=1}^{\infty}$ is the sequence of the compact sets as taken in the proof of Theorem 1. Now we set $u_n = \max\{1 - f_j \vee e_{X\setminus F_j}\,;\, 1 \leq j \leq n\}$, where $e_{X\setminus F_n}$ is an equilibrium potential of $X \setminus F_n$. Then this yields desired statement.

4. Application of Theorem 1. Shigekawa and Taniguchi [12] used the Gel'fand compactification to show the uniqueness statement for a measure charging no set of zero capacity. But they showed, without using any type of compactification, the following Beurling-Deny formula under the condition (A.1, 2', 3, 4). Based on Theorem 1, we can show it in the same as in [12] under the condition (A.1-4).

A finite positive Borel Measure μ on X charging no set of zero capacity is said to be *of finite energy integral* if there is a constant $C > 0$ such that

$$\int_X |f|\, d\mu \leq C\sqrt{\mathscr{E}_1(f, f)} \quad \text{for } f \in b\mathscr{F}_0.$$

Theorem 2. *Let* $(\mathscr{E}, \mathscr{F})$ *be a Dirichlet form which satisfies* (A.1-4). *Then* $\mathscr{E}$ *can be expressed for* f, $g \in \mathscr{F}$ *as follows.*

(4.1)

$$\mathscr{E}(f, g) = \mathscr{E}^{(c)}(f, g) + \int_{(X\times X)\setminus D} (f(x) - f(y))(g(x) - g(y))J(dx \times dy) + \int_X f(x)g(x)k(dx).$$

Here $\mathscr{E}^{(c)}$ *is a symmetric form with local property,* J *is a* σ*-finite symmetric measure* on $(X \times X)\setminus D$, *with* D *the diagonal set of* $X \times X$, *satisfying* $J(X \times A) = 0$ *if* $\mathrm{Cap}(A) = 0$, *and* k *is a finite positive Borel measure of finite energy integral. These* $\mathscr{E}^{(c)}$, J *and* k *are determined uniquely by* $\mathscr{E}$.

These are some other facts which can be shown with Theorem 1. Under the condition (A.1-4) there exists an associated Hunt process, and a hitting distribution is a version of a equilibrium potential. It can be seen by the Hunt approximation theorem that a nearly Borel, finely open and m-negligible set is exceptional and a set is exceptional if and only of it is included in a properly exceptional set. We can also show that if u is quasi-continuous, then u is finely continuous q.e.; conversely, if u is finely continuous q.e. and $u \in \mathscr{F}$. then u is quasi-continuous. Moreover, by Remark 1, we also see the following two conditions are equivalent to each other: (1) $(\mathscr{E}, \mathscr{F})$ possesses the local property; (2) the associated Hunt process have continuous sample paths with probability 1 (cf. [12, Theorem 6.1]).

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