52. *A Note on the Irrationality of Certain Infinite Series*

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 14, 1992)

1. Statement of result. Let $\{a_n\}$ be a sequence of positive integers satisfying the next three conditions:

(1) $a_1 \geqq 2$,

(2) $a_{n+1} \geqq a_n$ for all sufficiently large n,

(3) $\lim_{n\to\infty} a_n = \infty$.

We put

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{A_k}$$

and

$$\beta = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{A_k},$$

where A_k is defined by

$$A_k = \prod_{n=1}^{k} a_n.$$

The aim of this note is to prove the following theorem which includes the result of Iséki [1] as a special case.

Theorem. *The three numbers* 1, α *and* β *are linearly independent over the field of rational numbers.*

We shall complete the proof of the theorem by using the elementary method which was employed by Siegel [2] to show that e is not a quadratic irrationality.

2. Proof of the theorem. Let n be a sufficiently large integer to ensure the validity of the later argument.

We put $\alpha = \gamma_n + \delta_n$ and $\beta = \rho_n + \sigma_n$, where

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{A_k}, \ \delta_n = \sum_{k=n+1}^{\infty} \frac{1}{A_k}, \ \rho_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{A_k} \quad \text{and}$$

$$\sigma_n = \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{A_k}.$$

Further we put $C_n = A_n\gamma_n$, $D_n = A_n\delta_n$, $R_n = A_n\rho_n$ and $S_n = A_n\sigma_n$. Then we see that C_n and R_n are integers and that

$$0 < D_n < \frac{1}{a_{n+1}-1} \text{ and } 0 < (-1)^n S_n < \frac{1}{a_{n+1}-1}.$$

Let p and q denote arbitrary integers, not both 0.

Put $E_n = A_n(p\alpha + q\beta) = (pC_n + qR_n) + (pD_n + qS_n) = T_n + U_n$, say.

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Then it is easy to check that T_n is an integer and

$$|U_n| \leqq |pD_n| + |qS_n| \leqq \frac{|p|+|q|}{a_{n+1}-1} < 1.$$

As is easily seen,

$$a_n U_{n-1} - U_n = p(a_n D_{n-1} - D_n) + q(a_n S_{n-1} - S_n) = p + (-1)^{n-1} q,$$

so that at least one of the three numbers U_{n-1}, U_n and U_{n+1} is different from 0, since otherwise $p+q=0$, $p-q=0$ and $p=q=0$, which is a contradiction. This shows the existence of a positive integer ν such that E_ν is not integral. Therefore the number $\dfrac{E_\nu}{A_\nu} + r = p\alpha + q\beta + r$ is different from 0, for all integral r. This means that $p\alpha + q\beta + r \neq 0$ for arbitrary integers p, q and r, not all 0, which implies our assertion.

References

[1] K. Iséki: On the irrationality of the sum of some infinite series. Math. Sem. Notes Kobe Univ., **7**, 183–184 (1979).

[2] C. L. Siegel: Transcendental Numbers. Princeton Univ. Press (1949).