# 8. *Formation of Singularities in Solutions of the Nonlinear Schrödinger Equation**)

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§ 1. **Introduction and results.** This paper is a sequel to the previous ones [5] and [6]. We continue the study of the $L^2$-concentration in solutions of initial value problem for the nonlinear Schrödinger equation:

$$(\mathrm{Cp})\quad \begin{cases} (\mathrm{NLS}) & 2i\dfrac{\partial u}{\partial t}+\Delta u+|u|^{4/N}u=0, \qquad (t,x)\in \boldsymbol{R}^{+}\times \boldsymbol{R}^{N}, \\ (\mathrm{IV}) & u(0,x)=u_0(x), \qquad x\in \boldsymbol{R}^{N}, \end{cases}$$

where $i=\sqrt{-1}$, $u_0\in H^1=H^1(\boldsymbol{R}^N)$, $\Delta$ is the Laplacian on $\boldsymbol{R}^N$.

The local existence theory for (Cp) is well known ([1], [3]); there are $T_m\in(0,\infty]$ (maximal existence time) and a unique solution $u(\cdot)\in C([0,T_m);H^1)$ of (Cp). Furthermore $u$ satisfies

$$(1.1)\qquad \|u(t)\|=\|u_0\|,$$

$$(1.2)\qquad E(u(t))\equiv\|\nabla u(t)\|^2-(2/\sigma)\|u(t)\|_\sigma^\sigma=E(u_0),$$

for $t\in[0,T_m)$. Here $\sigma=2+4/N$ and $\|\cdot\|$ ($\|\cdot\|_\sigma$) denotes the $L^2(\boldsymbol{R}^N)(L^\sigma(\boldsymbol{R}^N))$-norm.

It is also well-known (see [2]) that, for some $u_0$, the solution $u$ shows the singular behavior (blow-up) that

$$(1.3)\qquad \lim_{t\to T_m}\|\nabla u(t)\|=\|u(t)\|_\sigma=\infty$$

for some $T_m\in(0,\infty]$.

Of physical importance is the case $N=2$, when (NLS) is a model of the stationary self-focusing of a laser beam propagating along the $t$-axis. It is considered that the singular behavior (1.3) corresponds to the focus of the beam. Thus our purpose is to obtain more precise analysis of the behavior of the singular solution $u(t)$ of (Cp) as $t\uparrow T_m$. Because of its mathematical interest however, we intend to develop a theory for arbitrary dimensions $N$. It should be noted that (NLS) has a remarkable property that it is invariant under the pseudo-conformal transformations.

In [6], we proved;

**Proposition A.** *Suppose that the solution $u(t)$ of (Cp) satisfies (1.3). Let $(t_n)_n$ be any sequence such that $t_n\to T_m$ as $n\to\infty$. Set*

$$(\mathrm{A.1})\qquad \lambda_n\equiv\lambda(t_n)=1/\|u(t_n)\|_\sigma^{\sigma/2}\qquad (\longrightarrow 0 \text{ as } n\longrightarrow\infty),$$

$$(\mathrm{A.2})\qquad u_n(t,x)\equiv S_{\lambda_n}u(t,x)=\lambda_n^{N/2}u(t,\lambda_n x).$$

*Then there exists a subsequence of $(t_n)_n$ (we still denote it by $(t_n)_n$) which satisfies the following properties: one can find $L\in\boldsymbol{N}\cup\{\infty\}$ and sequences $(y_n^j)_n$ in $\boldsymbol{R}^N$ for $1\leq j\leq L$ such that*

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*) In memory of my father.

(A.3) $\lim_{n\to\infty} |y_n^j - y_n^k| = \infty$ $(j \neq k)$,

(A.4) $f_n^1 \equiv u_n(t_n, x+y_n^1) \longrightarrow f^1$ *weakly in* $H^1$,

(A.5) $f_n^j \equiv (f_n^{j-1} - f^{j-1})(\cdot + y_n^j) \longrightarrow f^j$ *weakly in* $H^1$,

(A.6) $\lim_{n\to\infty} \{E(f_n^j) - E(f_n^j - f^j)\} = E(f^j)$,

(A.6)′ $\lim_{n\to\infty} E(f_n^j - f^j) = -\sum_{k=1}^{j} E(f^k)$,

(A.7) $\lim_{j\to L} \lim_{n\to\infty} \| f_n^j - f^j \|_\sigma = 0$ $(L = +\infty)$,

(A.7)′ $\lim_{n\to\infty} \| f_n^L - f^L \|_\sigma = 0$ $(L < +\infty)$,

(A.8) $\lim_{j\to L} \lim_{n\to\infty} \left\{ \sup_{y\in \boldsymbol{R}^N} \int_{B(y;R)} |(f_n^j - f^j)(x)|^2 dx \right\} = 0$ *if* $L = +\infty$,

(A.8)′ $\lim_{n\to\infty} \left\{ \sup_{y\in \boldsymbol{R}^N} \int_{B(y;R)} |(f_n^L - f^L)(x)|^2 dx \right\} = 0$ *if* $L < +\infty$,

*where $R$ is any positive constant and $B(y;R) = \{x \in \boldsymbol{R}^N; |x-y| \leqq R\}$.*

Using this proposition and the characterization of $Q$ (see (B.1) below), we also proved in [6]

**Theorem B.** *Let $Q$ be a ground state (non trivial minimal $L^2$ norm) solution of*

(B.1) $$\Delta Q - Q + |Q|^{4/N} Q = 0, \qquad Q \in H^1.$$

*Under the same assumptions and notations of Proposition A, then there exists a subsequence of $(t_n)_n$ (we still denote it by $(t_n)_n$) which satisfies the following properties: one can find a sequence $(y_n)_n$ in $\boldsymbol{R}^N$ such that, for any $\varepsilon > 0$, there is a positive constant $K$;*

(B.2) $$\liminf_{n\to\infty} \int_{B(R)} |S_{\lambda_n} u(t_n, x+y_n)|^2 dx \geqq (1-\varepsilon) \|Q\|^2$$

*for any $R \geqq K$. In other words,*

(B.3) $$\liminf_{n\to\infty} \int_{B_n} |u(t_n, x)|^2 dx \geqq (1-\varepsilon) \|Q\|^2,$$

*where $B_n = \{x \in \boldsymbol{R}^N; |x - y_n \lambda_n| \leqq R\lambda_n\}$ $(\forall R \geqq K)$.*

**Remarks.** (1) If $\|u_0\| < \|Q\|$, the corresponding solution $u(t)$ exists globally in time; $u(\cdot) \in C([0,\infty); H^1) \cap L^\infty(0,\infty; H^1)$. The initial datum $u_0 = Q(x) \exp(-i|x|^2/2)$ $(\|u_0\| = \|Q\|)$ leads to the solution $u(t)$ which satisfies (1.3) with $T_m = 1$ and $|u(t,x)|^2$ approaching to $\|Q\|^2 \delta(x)$ (Dirac measure) as $t \to 1$ (see [7] and [9]).

(2) The spatial dilation operator $S_\lambda$ was introduced by Weinstein for the first time in [9]. Our scaling function $\lambda$, however, is different from the one in [9].

In this paper, we extend Theorem B to show

**Theorem C.** *Suppose that the solution $u(t)$ of (Cp) satisfies (1.3). Set*

(C.1) $$\lambda(t) = 1/\|u(t)\|_\sigma^{\sigma/2},$$

(C.2) $$S_\lambda u(t,x) = \lambda^{n/2} u(t, \lambda x),$$

(C.3) $$A \equiv \sup_{R>0} \liminf_{t \uparrow T_m} \left\{ \sup_{y\in \boldsymbol{R}^N} \int_{B(y;R)} |S_{\lambda(t)} u(t,x)|^\sigma dx \right\}.$$

*If* $A=1$, *then, for any* $0<\varepsilon<1$, *there are constants* $K>0$, $T_0>0$ *and* $\gamma(\cdot)\in C([T_0, T_m); \boldsymbol{R}^N)$ *such that*

$$\text{(C.4)} \qquad \int_{B(R)} |S_{\lambda(t)}u(t, x+\gamma(t))|^2 dx > (1-\varepsilon)\|Q\|^2$$

*for any* $R\geqq K$. *In other words,*

$$\text{(C.5)} \qquad \int_{B_t} |u(t, x)|^2 dx > (1-\varepsilon)\|Q\|^2$$

*where* $B_t=\{x\in \boldsymbol{R}^N; |x-\gamma(t)\lambda(t)|\leqq R\lambda(t)\}$ $(\forall R\geqq K)$.

**Remarks.** (1) Suppose that $\|u_0\|=\|Q\|$ and corresponding solution $u(t)$ of (Cp) satisfies (1.3). Then we have $A=1$.

(2) Suppose that $u_0$ is radially symmetric, $N=2$ and corresponding solution $u(t)$ of (Cp) satisfies (1.3). Then we have $A=1$. In this case, we can take $\gamma\equiv 0$.

(3) The condition $A=1$ (see (C.3)) implies that $L=1$ in Proposition A for any sequence $t_n\to T_m$. We may regard $\gamma(t)$ in Theorem C as a "ray trajectory" for the beam described by the solution $u(t)$ of (Cp) with $A=1$.

**§2. Proof of Theorem C.** Suppose that the solution $u(t)$ to (Cp) satisfies (1.3) and

$$\text{(2.1)} \qquad 1=\sup_{R>0}\ \liminf_{t\uparrow T_m}\left\{\sup_{y\in \boldsymbol{R}^N}\int_{B(y;R)} |S_{\lambda(t)}u(t, x)|^\sigma dx\right\}.$$

For simplicity, we suppose $N\geqq 3$. We will use the notations;

$$B_y=B(y; R)=\{x\in \boldsymbol{R}^N; |x-y|\leqq R\}, \qquad B_{y(t)}=B(y(t); R),$$
$$u_\lambda(t, x)=S_{\lambda(t)}u(t, x),$$
$$P_\sigma(t; \Omega)=\int_\Omega |u_\lambda(t, x)|^\sigma dx \qquad \text{for any } \Omega\subset \boldsymbol{R}^N.$$

We recall that $\lambda\equiv\lambda(t)=1/\|u(t)\|_\sigma^{\sigma/2}$. One can see that

$$\text{(2.2)} \qquad \|u_\lambda\|=\|u\|=\|u_0\|, \qquad \|u_\lambda\|_\sigma=1.$$

Moreover we have that

$$\text{(2.3)} \qquad E(u_{\lambda(t)})=\lambda^2(t)E(u(t))=\lambda^2(t)E(u_0)\longrightarrow 0$$

as $t\to T_m$. From (2.2), (2.3) and Sobolev's inequality, one has

$$\text{(2.4)} \qquad \|u_\lambda\|_{2*}\leqq S\|\nabla u_\lambda\|\leqq S$$

for sufficiently small $\lambda$, where $S$ is the Sobolev best constant and $\|\cdot\|_{2*}$ denotes the $L^{2N/(N-2)}$-norm.

We start with

**Proposition 2.1.** *For any* $0<\varepsilon<1$, *there are constants* $K>0$, $T_0>0$ *and a function* $\gamma(\cdot)\in C([T_0, T_m); \boldsymbol{R}^N)$ *such that*

$$\text{(2.5)} \qquad \int_{B(R)} |u_\lambda(t, x+\gamma(t))|^\sigma dx > 1-\varepsilon, \qquad t\in[T_0, T_m),$$

*for any* $R\geqq K$.

For the proof of this proposition, we prepare

**Lemma 2.2.** *Let* $y_*$ *be a point such that* $P_\sigma(T_*; B(y_*; R))>1-\varepsilon/2$ *holds true at a time* $T_*\in[0, T_m)$ *for some constant* $R>0$. *Then there exist positive constants* $\theta$ *and* $\Gamma$ *such that if* $|t-T_*|<\theta$ *and* $|y_*-y|<\Gamma$, *then* $P_\sigma(t; B(y; R))>1-\varepsilon/2$.

*Proof of Lemma* 2.2. Let $A'=P_\sigma(T_*; B(y_*; R))$ and $B_*=B_{y_*}$, and put

(2.6) $$3\varepsilon'=A'-(1-\varepsilon/2).$$

We note that

(2.7) $$P_\sigma(T_*;B_*\cap B_y)+P_\sigma(T_*;B_*-B_y)=P_\sigma(T_*;B_*)=A',$$

for any $y\in \boldsymbol{R}^N$. For $\varepsilon'>0$ defined in (2.6), there is a positive constant $\Gamma$ such that if $|y_*-y|<\Gamma$, then it holds for any $t$ that

(2.8) $$P_\sigma(t;B_y-B_*)<\varepsilon',$$

since we have, by Hölder's inequality and (2.4)

$$P_\sigma(t;B_y-B_*)^{1/2}\leqq\mu(B_y-B_*)^{2/N}\|u_\lambda\|_{2^*}^{\sigma/2}\leqq S\mu(B_y-B_*)^{2/N}.$$

On the other hand, since $u_\lambda\in C([0,(T_*+T_m)/2];L^2)$ (uniformly continuous in $t$), there exists a positive constant $\theta$ such that if $|T_*-t|<\theta$, one has

(2.9) $$-\varepsilon'+P_\sigma(T_*;B_y\cap B_*)<P_\sigma(t;B_y\cap B_*)$$

(2.10) $$-\varepsilon'+P_\sigma(T_*;B_y-B_*)<P_\sigma(t;B_y-B_*).$$

Here we note that $\theta$ depends on $T_*$. Therefore if $|T_*-t|<\theta$ and $|y_*-y|<\Gamma$, we have, adding (2.9) and (2.10),

(2.11) $$\begin{aligned}P_\sigma(t;B_y)&>P_\sigma(T_*;B_y)-2\varepsilon'\\&=P_\sigma(T_*;B_y\cap B_*)+P_\sigma(T_*;B_y-B_*)-2\varepsilon'\\&\geqq A'-P_\sigma(T_*;B_*-B_y)+P_\sigma(T_*;B_y-B_*)-2\varepsilon'.\end{aligned}$$

Here we have used (2.7). By (2.6), (2.8) and (2.11), we obtain

(2.12) $$P_\sigma(t;B_y)>A^1-3\varepsilon'>1-\varepsilon/2,$$

if $|T_*-t|<\theta$ and $|y-y_*|<\Gamma$.

*Proof of Proposition* 2.1. We have by the definition (2.1) that, for any $\varepsilon>0$, there exist $K>0$, $T_0>0$ and $y(t)\in\boldsymbol{R}^N$ for $t\in[T_0,T_m)$ such that

(2.13) $$P_\sigma(t;B(y(t);R))>1-\varepsilon/2,\quad t\in[T_0,T_m),\quad R\geqq K.$$

We define

$$T^*=\sup\{T\in[T_0,T_m);\ P_\sigma(T;B(y(T_0);R))>1-\varepsilon/2\}.$$

By Lemma 2.2, $T^*>T_0$. If $T^*=T_m$, nothing to prove. We suppose $T^*<T_m$. On the other hand, we have by Lemma 2.2,

(2.14) $$P_\sigma(t;B(y(T^*);R))>1-\varepsilon/2,\qquad t\in[T^*-\theta,T^*]$$

for some $\theta>0$. For brevity, we put $I^*=[T^*-\theta,T^*]$, $y^*=y(T^*)$, $y_*=y(T_0)$, $B^*=B(y^*;R)$ and $B_*=B(y_*;R)$.

**Claim 1.** *$(B^*\times\{t\})\cap(B_*\times\{t\})\neq\emptyset$ for any $t\in I^*$.*

*Proof.* Suppose that $(B^*\times\{t\})\cap(B_*\times\{t\})=\emptyset$ for some $t\in I^*$. Then we have, by the definition of $T^*$ and (2.14).

$$1=\|u_\lambda\|_\sigma^\sigma\geqq P_\sigma(t;B^*)+P_\sigma(t;B_*)>(1-\varepsilon/2)+(1-\varepsilon/2)=(2-\varepsilon)$$

for $t\in I^*$, so that we get $(1-\varepsilon)<0$. Thus we reach a contradiction.

**Claim 2.** $P_\sigma(t;B^*\cap B_*)>1-\varepsilon$, $t\in[T^*-\theta,T^*)$.

*Proof.* We have, by (2.14), the definition of $T^*$ and the above claim,

$$\begin{aligned}1&=\|u_\lambda\|_\sigma^\sigma\geqq P_\sigma(t;B^*\cup B_*)\\&=P_\sigma(t;B^*)+P_\sigma(t;B_*)-P_\sigma(t;B^*\cap B_*)>(2-\varepsilon)-P_\sigma(t;B^*\cap B_*).\end{aligned}$$

Thus one has

$$P_\sigma(t;B^*\cap B_*)>1-\varepsilon,\qquad t\in[T^*-\theta,T^*).$$

Now we define

(2.15) $$\begin{cases}\gamma(t)=y_*, & t\in[T_0,T^*-\theta)\\ \gamma(t)=y^*+\{(T^*-t)/\theta\}(y_*-y^*), & t\in[T^*-\theta,T^*).\end{cases}$$

One can easily see that

(2.16) $\gamma(\cdot)\in C([T_0, T^*]; \boldsymbol{R}^N)$,

(2.17) $P_\sigma(t; B(\gamma(t); R))>1-\varepsilon, \quad t\in[T_0, T^*]$

by Claim 2 and (2.14), since $B(\gamma(t); R)\supset B^*\cap B_*$.

We note that there is a positive constant $\theta'$ $(<\theta)$ such that

(2.18) $P_\sigma(t; B(\gamma(t); R))>1-\varepsilon/2, \quad t\in[T^*-\theta', T^*]$

by Lemma 2.7.

Hence repeating the above argument starting with $y^*$ instead of $y_*$, we can obtain a *continuous path* $\gamma(t)$; $[T_0, T_m)\to \boldsymbol{R}^N$ which satisfies (C.4).

To conclude the proof of Theorem C, we must show the following lemma for the "path" $\gamma(t)$ constructed in Proposition 2.1.

**Lemma 2.3.** *There are constants* $K_1>0$, $T_1>0$ *such that*

(2.19) $$\int_{B(R)}|u_\lambda(t, x+\gamma(t))|^2dx>(1-\varepsilon)\|Q\|^2, \qquad t\in[T_1, T_m),$$

*for any* $R\geqq K_1$.

*Proof.* Suppose the contrary, so that, any $n\in\boldsymbol{N}$, there are $R_n\geqq n$ and $t_n\in(T_m-1/n, T_m)$ such that

(2.20) $$\int_{B(R_n)}|u_\lambda(t_n, x+\gamma(t_n))|^2dx\leqq(1-\varepsilon)\|Q\|^2.$$

According to this sequence $(t_n)_n$, we put $u_n^1(x)\equiv u_{\lambda_n}(t_n, x+\gamma(t_n))$.

On the other hand, by virtue of the first concentration-compactness lemma due to Lions (see [4; Appendix]) together with (2.1) and the latter of (2.2), we can find a sequence $(y_n)_n$ in $\boldsymbol{R}^N$ for the above $(t_n)_n$ such that for any $\eta>0$,

(2.21) $$1>\int_{B(R)}|u_{\lambda_n}(t_n, x+y_n)|^\sigma dx>1-\eta,$$

for sufficiently large $R>0$ and $n$. We put $f_n^1(x)\equiv u_{\lambda_n}(t_n, x+y_n)$.

Then $(u_n^1)_n$ and $(f_n^1)_n$ are bounded sequence in $H^1$ and they converges weakly to non trivial elements in $H^1$, since we have (2.5) and (2.21). This is valid only for a subsequence. We shall often extract subsequence without explicitly mentioning this fact. Since $\eta>0$ is arbitrary, $f_n^1$ converges to $f\in H^1$ strongly in $L^\sigma$ by the latter of (2.2). One can easily see that $\sup_{n\geqq1}|\gamma(t_n)-y_n|<\infty$ by (2.5) and (2.21), so $u_n^1$ also converges to $u^1\in H^1$ strongly in $L^\sigma$. This corresponds to the case $L=1$ in Proposition A. Thus we have $E(u^1)\leqq0$ by (A.6) and (2.3), so that $\|u^1\|\geqq\|Q\|$ follows from the characterization of $Q$ (see e.g. [6; Lemma 1.1]). Therefore letting $n\to\infty$ in (2.20) (using Fatou's lemma), we reach a contradiction.

**§ 3. Generalizations.** The nonlinear term $|u|^{4/N}u$ can be replaced by the more general one $F(u)$ treated in [5] and [6]; typical examples of $F$ are

(NF) $F(u)=|u|^{4/N}u+\chi|u|^{q-1}u, \quad \chi\in\boldsymbol{R}, \quad 1\leqq q<1+4/N.$

For generic blow-up solution, using Proposition A and the argument performed in [5], we can prove

**Theorem D.** *Suppose that the solution* $u(t)$ *to* (*Cp*) *with the nonlinear term* (*NF*) *satisfies* (1.3). *Set*

(D.1) $\lambda(t)=1/\|u(t)\|_{\sigma}^{\sigma/2}$,

(D.2) $S_{\lambda}u(t,x)=\lambda^{n/2}u(t,\lambda x)$,

(D.3) $$A\equiv\sup_{R>0}\liminf_{t\uparrow T_m}\left\{\sup_{y\in\boldsymbol{R}^N}\int_{B(y;R)}|S_{\lambda(t)}u(t,x)|^2dx\right\}.$$

*Then we have* $A\geqq\|Q\|^2$ *and, for any* $0<\varepsilon<1$, *there are constants* $K>0$, $T_0>0$ *and a right continuous function* $y\in L^{\infty}_{\text{loc}}([T_0,T_m);\boldsymbol{R}^N)$ *such that*

(D.4) $$\int_{B(R)}|S_{\lambda(t)}u(t,x+y(t))|^2dx>(1-\varepsilon)A,\qquad t\in[T_0,T_m),$$

*for any* $R\geqq K$.

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## References

[1] Ginibre, J. and Velo, G.: On a class of nonlinear Schrödinger equations. I, II. J. Funct. Anal., **32**, 1–71 (1979).

[2] Glassy, R. T.: On the blowing up solutions to the Cauchy problem for nonlinear Schrödinger equations. J. Math. Phys., **18**, 1794–1797 (1979).

[3] Kato, T.: On nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Physique Theorique., **46**, 113–129 (1987).

[4] Lions, P. L.: Solutions of Hartree-Fock equations for coulomb systems. Commun. Math. Phys., **109**, 33–97 (1987).

[5] Nawa, H.: "Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity (1989) (preprint).

[6] ——: "Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity. II. Kodai Math. J., **13**, 333–348 (1990).

[7] Nawa, H. and Tsutsumi, M.: On blow-up for the pseudo-conformally invariant nonlinear Schrödinger equation. Funk. Ekva., **32**, 417–428 (1989).

[8] Weinstein, M. I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys., **87**, 567–576 (1983).

[9] ——: On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations. Comm. in Partial Differential Equations, **11**, 545–565 (1986).