68. *Generalized Interface Evolution with the Neumann Boundary Condition*

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1. Introduction. We are concerned with geometric evolution (e.g. motion by mean curvature) of interfaces in a smoothly bounded domain Ω ($\subset \boldsymbol{R}^n$) whose boundary $\partial\Omega$ perpendicularly intersects with interfaces. In [12] the second author extended a level set approach introduced by Chen-Giga-Goto [1] and Evans-Spruck [3] to this type of the Neumann problem and obtained a unique global weak solutions for the initial value problem provided that Ω is convex. This note reports that the convexity assumption of Ω can be removed. The details and proofs will appear elsewhere.

One of key ingredients is the comparison principle for the Neumann boundary value problem for singular degenerate parabolic equations. For the Neumann problem this principle is first established by Lions [10] for the Hamilton-Jacobi equations. For nonsingular degenerate elliptic equations the comparison principle is established by Ishii and Lions [9]. See also [8] for more general oblique boundary conditions. However, their argument does not apply to singular equations. In [12] the second author obtained the comparison principle for our problem assuming that Ω is convex. His method appeals to the idea of [6] by regarding $\partial\Omega$ as space infinity. Unfortunately, the choice of test functions does not apply to general domains. In this note we construct test functions by using local coordinate patches near $\partial\Omega$ so that they apply to general domains.

In [7] Huisken considers the interface intersecting perpendicularly with $\partial\Omega$ and moving by mean curvature. He constructed a global smooth evolution of interfaces when Ω is a cylindrical domain $D\times \boldsymbol{R}$ and the initial interface is the graph of a smooth function on D, where D is bounded. Although our theory presented below assumes that Ω is bounded, it can be extended to cylindrical domain $D\times \boldsymbol{R}$ provided D is bounded. The motion by mean curvature with right contact angle at $\partial\Omega$ arises as a singular limit of a reaction-diffusion equation with the Neumann condition [11].

2. Comparison principle. We here present a simple and typical version of our comparison principle rather than stating its general form to avoid technical complexity. We consider an evolution equation of the form

(1) $$u_t+F(\nabla u, \nabla^2 u)=0 \quad \text{in } Q=(0,T)\times\Omega$$

(2) $$\partial u/\partial\nu=0 \quad \text{on } S=(0,T)\times\partial\Omega,$$

where $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$; $u_t=\partial u/\partial t$, $\nabla u=\operatorname{grad} u$; $\nabla^2 u$ denotes the Hessian of u in the space variables. We list assumptions on F.

(F1) $F:(\boldsymbol{R}^n\backslash\{0\})\times\boldsymbol{S}^n\to\boldsymbol{R}$ is continuous, where $\boldsymbol{S}^n$ denotes the space of $n\times n$ real symmetric matrices equipped with usual ordering.

(F2) F is degenerate elliptic, i.e., $F(p,X+Y)\leq F(p,X)$ for all $Y\geq 0$.

(F3) $-\infty<F_*(0,O)=F^*(0,O)<\infty$ where F_* and F^* are the lower and upper semicontinuous relaxation (envelope) of F on $\boldsymbol{R}^n\times\boldsymbol{S}^n$, respectively, i.e.,

$$F_*(p,X)=\liminf_{\varepsilon\downarrow 0}\{F(q,Y);\ q\neq 0,\ |p-q|\leq\varepsilon,\ |X-Y|\leq\varepsilon\}$$

and $F^*=-(-F)_*$. Here $|X|$ denotes the operator norm of X as a self adjoint operator on $\boldsymbol{R}^n$.

Theorem 1. *Let Ω be a smoothly bounded domain in $\boldsymbol{R}^n$. Suppose that F satisfies* (F1)–(F3). *Let u and v be, respectively, viscosity sub- and supersolutions of* (1)–(2). *If $u^*(0,x)\leq v_*(0,x)$, then $u^*\leq v_*$ on $[0,T)\times\bar{\Omega}$.*

A definition of a viscosity (sub) solution for the Neumann problem goes back to [10] where the Hamilton-Jacobi equation is studied. We recall a definition of viscosity subsolution of (1)–(2) for the reader's convenience. We refer to [2] and [8] for nonsingular equations. Any function $u: Q\cup S\to\boldsymbol{R}$ is called a *viscosity subsolution* of (1)–(2) if $u^*<\infty$ on $\bar{Q}$ and if, whenever $\phi\in C^2(Q\cup S)$, $(t,x)\in Q\cup S$ and $(u^*-\phi)(t,x)=\max_{Q\cup S}(u^*-\phi)$, one of the following holds

$$(3)\qquad \phi_t(t,x)+F_*(\nabla\phi(t,x),\nabla^2\phi(t,x))\leq 0$$

$$(4)\qquad (\partial\phi/\partial\nu)(t,x)\leq 0 \quad\text{and}\quad x\in\partial\Omega.$$

For example a function $u(t,x)=-2t-|x|^2$ a viscosity subsolution (actually solution) of (1)–(2) with

$$(5)\qquad F(p,X)=-\operatorname{trace}((I-p\otimes p/|p|^2)X)$$

on an annulus Ω in $\boldsymbol{R}^2$ although $\partial u/\partial\nu\leq 0$ may not hold on the inner circle of $\partial\Omega$ in usual sense. One should be careful with the meaning of (2).

3. **Test functions.** The basic strategy of the proof of Theorem 1 is to find a parabolic super 2-jet of

$$w(t,x,y)=u(t,x)-v(t,y)$$

at a point where $u^*>v_*$. This idea is the same as in [6] and we also apply the Crandall-Ishii lemma (see e.g. [2]). Since it is difficult to compare boundary condition (4), we take a barrier near the boundary to avoid to handle (4). This idea is found in [12].

For $\varepsilon,\delta,\gamma>0$ we set

$$\Phi(t,x,y)=w(t,x,y)-\Psi(t,x,y)$$

$$\Psi(t,x,y)=\Xi(x,y)/\varepsilon+B(t,x,y)$$

$$B(t,x,y)=\delta(\varphi(x)+\varphi(y)+2\beta)+\gamma/(T-t).$$

Here $\varphi\in C^2(\bar{\Omega})$ is a 'barrier' function of $\partial\Omega$ satisfying:

$$-\beta\leq\varphi<0 \text{ in } \Omega,\ \varphi=0 \text{ on } \partial\Omega \text{ with a constant } \beta>0$$

$$\nu(x)=\nabla\varphi(x)/|\nabla\varphi(x)| \quad\text{and}\quad |\nabla\varphi(x)|\geq 1 \quad\text{on } \partial\Omega.$$

If $\Xi \in C^2(\bar{\Omega}\times\bar{\Omega})$ satisfies following conditions, the method of [6] applies to establish Theorem 1 by using (3).

(C1) $\Xi(x,y)\geq c_0|x-y|^4$ with $c_0>0$.

(C2) $|\Xi_x+\Xi_y|\leq c_1|x-y|^4$, $|\Xi_x|$, $|\Xi_y|\leq c_2|x-y|^3$.

(C3) $|\Xi_{xx}+\Xi_{xy}+\Xi_{yx}+\Xi_{yy}|\leq c_3|x-y|^4$.

(C4) $|\Xi_{xx}|$, $|\Xi_{xy}|$, $|\Xi_{yx}|$, $|\Xi_{yy}|\leq c_4|x-y|$.

(C5) $\langle \nu(x), \Xi_x(x,y)\rangle \geq 0$ for $x\in\partial\Omega$, $y\in\bar{\Omega}$

$\langle \nu(y), -\Xi_y(x,y)\rangle \leq 0$ for $y\in\partial\Omega$, $x\in\bar{\Omega}$

provided that $|x-y|$ is sufficiently small.

Here $\langle\ ,\ \rangle$ denotes the inner product in $\boldsymbol{R}^n$. If Ω is convex, then $\Xi(x,y)=|x-y|^4$ satisfies (C1)–(C5). However, for nonconvex Ω, this choice of Ξ violates (C5).

Lemma 2. *There exists Ξ satisfying* (C1)–(C5).

Sketch of the proof. For each $a\in\partial\Omega$ there is a local coordinate $\chi_a=(\chi^1,\cdots,\chi^n)$ such that $\chi^n(x)=\mathrm{dist}(x,\partial\Omega)$ for $x\in\bar{\Omega}$. Let ψ_a be a cut-off function supported near $a\in\partial\Omega$ so that $\partial\psi_a/\partial\nu=0$ on $\partial\Omega$. We set

$$\Lambda_a(x,y)=\psi_a(x)\psi_a(y)|\chi_a(x)-\chi_a(y)|^4.$$

Let ψ_0 be a cut-off function supported outside the boundary. We set

$$\Lambda_0(x,y)=\psi_0(x)\psi_0(y)|x-y|^4.$$

One can take finitely many $\{a_k\}_{k=1}^l$ so that the sum $\sum_{k=0}^l \Lambda_k$ satisfies (C1)–(C5) provided that $|x-y|$ is sufficiently small. Here $\Lambda_k=\Lambda_a$ with $a=a_k$. We set

$$\Xi(x,y)=\rho(|x-y|)|x-y|^4+(1-\rho(|x-y|))\sum_{k=0}^{l}\Lambda_k(x,y)$$

with a cut-off function $\rho(\sigma)$ supported away from $\sigma=0$. One observes that Ξ satisfies (C1)–(C5).

4. Interface evolution. We remark that the theory in [1] and [4] can be extended to the motion of interfaces intersecting perpendicularly with $\partial\Omega$. The next lemma is fundamental to establish global solution for the initial value problem of (1)–(2) by Perron's method.

Lemma 3 ([12]). *Assume the hypotheses of Theorem* 1 *concerning F. Suppose that F is geometric. Then for $u_0\in C(\bar{\Omega})$ there are viscosity sub- and supersolutions u_-, u_+ of* (1)–(2) *with $u_{-*}(0,x)=u_+^*(0,x)=u_0(x)$.*

Although our theory applies to general interface equations as in [4], we state our results only for the motion by mean curvature just for simplicity.

Theorem 4. *Let D_0 be an open set in Ω. Let $u_0\in C(\bar{\Omega})$ satisfy $D_0=\{x\,;u_0(x)>0\}$. There is a unique viscosity solution $u\in C([0,\infty)\times\bar{\Omega})$ for* (1)–(2) *with* (5) *for arbitrary $T>0$ such that $u(0,x)=u_0(x)$. The set $D=\{(t,x)\,;u(t,x)>0\}$ is determined by D_0 and called a generalized evolution by mean curvature with initial data D_0 and the right angle boundary condition.*

Remark 5. In [4] D is determined by D_0 and $\Gamma_0=\{u_0(x)=0\}$. It turns out D is completely determined by D_0 as shown in [5].

Remark 6. If we take Ξ as sketched in §3, we need $C^{3,1}$ regularity of

$\partial\Omega$. However, by taking χ more clever way, we only need C^2 regularity of $\partial\Omega$ to establish Lemma 2.

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