66. *Selberg Zeta Functions and Ruelle Operators for Function Fields*

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§ 1. **Introduction.** Nowadays Selberg zeta functions are defined for semi-simple Lie groups G of real rank one and its discrete subgroups Γ, since Selberg defined it first in [11] in 1956. Two ways are known for studying Selberg zeta functions. One is by Selberg trace formulas and the other is by Ruelle operators. By the former method, Selberg zeta functions are finally expressed as the determinants of the Laplacians ([10], [2], [7], [6]). By the latter method, they are expressed as the determinants of **1**-(Ruelle operator) ([9], [3], [8]). Above all, the result of Mayer [8] is remarkable in number theoretic viewpoint. He treats the Selberg zeta function of $\Gamma=PSL(2, \boldsymbol{Z})$, which is the unique example of non-compact $\Gamma\backslash G$ for which the second method is applied.

In this paper we fulfill the second method in the case $\Gamma=PGL(2, \boldsymbol{F}[T])$, where $\boldsymbol{F}$ is a finite field of order q of odd characteristic. As is well be seen in number theory, Γ has similar properties to those of $PSL(2, \boldsymbol{Z})$. We can apply the second method by following Mayer [8]. For the present Γ, Akagawa [1] constructs Selberg trace formulas and proves that Selberg zeta functions are rational with respect to q^s. This paper will give another proof of Akagawa's result. It is much shorter than the original one, as is seen in the case of $PSL(2, \boldsymbol{Z})$ by Mayer [8]. In the next section, we will define continued fractions in function fields and deduce some properties. In the third section, we will classify conjugacy classes of Γ following Akagawa [1]. In the last section, we will define Ruelle operators on function spaces over function fields, and establish the relation to Selberg zeta functions.

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§ 2. **Continued fractions.** Throughout this paper, we fix q a power of an odd prime and $\boldsymbol{F}$ the finite field of order q. We denote R, K, and F to be $\boldsymbol{F}[T]$, $\boldsymbol{F}(T)$, and $\boldsymbol{F}((T^{-1}))$, which are analogues of $\boldsymbol{Z}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$. The field F is the completion of K with respect to the place T^{-1}. Elements in F are expressed as $x=\sum_{i=-\infty}^{k} a_i T^i$ ($a_i \in \boldsymbol{F}$, $k \in \boldsymbol{Z}$, $a_k \neq 0$). We will write deg for the map from F to $\boldsymbol{Z}$ such that the element x corresponds to k. The map deg is a homomorphism from F^* onto Z. When $\deg x \geq 0$, Gauss symbol [ ] is defined by $[x]:=\sum_{i=0}^{k} a_i T^i \in R$. In the following we assume $\deg x < 0$. We define Gauss map T as $Tx:=1/x-[1/x]$. Here $1/x$ is the inverse of x in F. It is easy to see that $\deg(Tx)<0$. Letting x_i be the polynomial $[1/T^i x]$, we have an expansion

$$x=\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{x_3+\cdots}}}.$$

Let $p_n, q_n \in R$ be polynomials which express the finite part of the above as

$$\frac{p_n}{q_n}=\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{\ddots}{+\cfrac{1}{x_{n-1}}}}}.$$

In what follows non-Archimedean absolute values in F are normalized as $|x|:=q^{\deg(x)}$. The following lemma says that p_n/q_n approximates x well.

Lemma 1.

$$\left|x-\frac{p_n}{q_n}\right|<\left|\frac{1}{q_n^2}\right|<\frac{1}{q^{2(n-1)}}\to 0 \quad (n\to\infty).$$

As a corollary of the above, we get the following.

Proposition. *Let $x \in F$ be an algebraic element over K. The following* (a) *and* (b) *are equivalent.*

(a) *x is quadratic over K.*

(b) *x has a cyclic expansion as continued fractions.*

§ 3. Conjugacy classes. The group $\Gamma=PGL(2,R)$ acts on the algebraic closure of K by the linear fractional transformation. Let $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ be a stabilizer of an algebraic element z. Then

$$z=\frac{a-d\pm\sqrt{(a+d)^2-4\alpha}}{2c},$$

where $\alpha=\det\gamma \in F^*$. We call z *real* or *imaginary* when it belongs to F or not, respectively. We call γ *hyperbolic* when its fixed point z is real quadratic. We can see that γ being hyperbolic is equivalent to that $\sqrt{(\operatorname{tr}\gamma)^2-4\alpha}$ is real. For any $p \in R$, $\sqrt{p}$ is real if and only if the degree of p is even and the coefficient of the term of the highest degree belongs to F^2. So we see that any element $\gamma \in \Gamma$ whose trace has a positive degree is hyperbolic. As trace depends only on the conjugacy class of γ, we get a correspondence between the set of hyperbolic conjugacy classes $[\gamma]$ and the set of real quadratic fields $K(z)$ over K. Let Prim(Γ) be the set of all the primitive hyperbolic conjugacy classes of Γ, where we call a class *primitive* when it cannot be expressed by a non-trivial power of any other conjugacy class. From the fact that γ^n $(n \in Z)$ has common fixed points, we establish a correspondence between $\mathrm{Prim}(\Gamma)$ and the set of quadratic fields;

$$\mathrm{Prim}(\Gamma)\ni[\gamma]\longmapsto K(\sqrt{(\mathrm{tr}\,\gamma)^2-4\det(\gamma)}).$$

Putting λ to be the eigenvalue of γ which has the larger non-archimedean absolute value, we define the norm of γ by $N(\gamma):=|\lambda|^2$. It is equal to $q^{2\deg\lambda}=q^{2\deg(\mathrm{tr}\,\gamma)}$. Selberg zeta function of Ruelle type is defined by

$$\zeta_\Gamma(s):=\prod_{\gamma\in\mathrm{Prim}(\Gamma)}(1-N(\gamma)^{-s})^{-1}.$$

Akagawa [1] constructs the Selberg trace formula and introduces the zeta function from the hyperbolic terms in the trace formula, which is

$$Z_\Gamma(s):=\prod_{\gamma\in\mathrm{Prim}(\Gamma)}(1-N(\gamma)^{-s})=\zeta_\Gamma(s)^{-1}.$$

Theorem ([1]). *Selberg zeta function $Z_\Gamma(s)$ absolutely converges in $\Re s>1$ and*

$$Z_\Gamma(s)=\frac{q^{2s}-q^2}{q^{2s}-q}.$$

§4. Ruelle operators. We define operators acting on the space $A(D)$ of analytic functions on $D=\{z\in F\,;\deg z<0\}$ into D;

$$\mathcal{L}_s:f(z)\longmapsto\sum_{n\in R-F}\left(\frac{1}{z+n}\right)^{2s}f\left(\frac{1}{z+n}\right),$$

where s is an integer. We see that the above converges for sufficiently large s. By the theorem of Goss [4, 2.1.2], values at large integers determine the function. By putting

$$\phi_n(z):=\left(\frac{1}{z+n}\right)^{2s},\quad\text{and}\quad T_n(z):=\frac{1}{z+n},$$

we decompose it into $\mathcal{L}_sf(z)=\sum_n\mathcal{L}_{s,n}f(z)$, where $\mathcal{L}_{s,n}f(z)=\phi_n(z)f(T_n(z))$. For a while we will fix s, and $\mathcal{L}_s$, $\mathcal{L}_{s,n}$ will be written as $\mathcal{L}$, $\mathcal{L}_n$. For given $x_i\in R$ $(i=1,2,3,\cdots)$, $[x_1,\cdots,x_n]$ will denote the cyclic continued fraction whose period is

$$\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{\ddots}{\ +\cfrac{1}{x_n}}}}.$$

Lemma 2.

$$\mathrm{tr}(\mathcal{L})=\sum_{n\in R-F}\frac{[n]^{2s}}{1+[n]^2}.$$

Proof. By the method of Kamowitz [5], we deduce that the spectrum of $\mathcal{L}_n$ consists only of eigenvalues, which are $\lambda_l=[n]^{2s}(-[n])^l$ $(l=0,1,2,\cdots)$. Q.E.D.

By the standard method of computing the trace of iterated Ruelle operators, the following is deduced.

Lemma 3.

$$\mathrm{tr}(\mathcal{L}^n)=\sum_{i_1,\cdots,i_n\in R-F}\frac{\prod_{k=1}^n[i_k,\cdots,i_n,i_1,\cdots,i_{k-1}]^{2s}}{1-(-1)^n\prod_{k=1}^n[i_k,\cdots,i_n,i_1,\cdots,i_{k-1}]^2}.$$

Proof. By the same method as that of Lemma 2, we can compute that the eigenvalues of $\mathcal{L}_{i_1}\mathcal{L}_{i_2}\cdots\mathcal{L}_{i_n}$ are given by

$$\lambda_l=\prod_{k=1}^{n}[i_k,\cdots,i_n,i_1,\cdots,i_{k-1}]^{2s}\prod_{k=1}^{n}(-[i_k,\cdots,i_n,i_1,\cdots,i_{k-1}]^2)^l. \quad \text{Q.E.D.}$$

We call x a *primitive* fixed element of T^n if x is not fixed by T^k for any positive integer $k<n$. The set of all the primitive fixed elements of T^n will be denoted by Prim (Fix T^n).

Lemma 4. *Let* $x=[i_1,\cdots,i_n]\in\text{Prim}(\text{Fix }T^n)$ *be a quadratic element over K, which corresponds to* $\gamma\in\Gamma$. *Then*

$$N(\gamma)=q^{2\sum_{k=1}^{n}\deg i_k}.$$

Let X be an operator having only eigenvalues as its spectrum. We write $\det(1-|X|)$ and $\text{tr}|X|$ for $\prod_\lambda(1-|\lambda|)$ and $\sum_\lambda|\lambda|$, respectively, when they absolutely converge.

Theorem.

$$\zeta_\Gamma(s)=\frac{\det(1-|\mathcal{L}_{s+1}|)}{\det(1-|\mathcal{L}_s|)}.$$

Proof. Using the equality $\det(1-|X|)=\exp-\{\sum_{m=1}^{\infty}(\text{tr}|X^m|)/m\}$, we can compute the right hand side by substituting Lemma 3. It is related to $\zeta_\Gamma(s)$ by Lemma 4. Q.E.D.

Corollary 1.

$$\zeta_\Gamma(s)=\frac{q^{2s}-q}{q^{2s}-q^2}.$$

Proof. From Lemma 3, $\text{tr}|\mathcal{L}^n|=\sum_{d=n}^{\infty}\varepsilon_n(d)(q^{-2sd}/(1-q^{-2d}))$, where $\varepsilon_n(d):=\#\{(i_1,\cdots,i_n)\in(R-F)^n\,|\,\sum_{l=1}^{n}\deg(i_l)=d\}$. It is computed that

$$\varepsilon_n(d)=\frac{q^d(q-1)^n(d-1)!}{(d-n)!\,(n-1)!}$$

and some technical calculations lead to the result. Q.E.D.

Corollary 2. *The Selberg zeta function* $\zeta_\Gamma(s)$ *satisfies a functional equation*

$$\zeta_\Gamma(s)\zeta_\Gamma\left(\frac{3}{2}-s\right)=\frac{1}{q}.$$

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