41. *On Non-stationary Boussinesq Equations*

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Let Ω be a bounded domain in $R^2(2\leq n\leq 4)$, the boundary of which satisfies the next condition.

Condition (H). *$\partial\Omega$ is of class C^1 and divided as follows: $\partial\Omega=\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2=\phi$, measure of $\Gamma_1\neq 0$, and the intersection $\bar{\Gamma}_1\cap\bar{\Gamma}_2$ is an $n-2$ dimensional C^1 manifold.*

We consider the following initial boundary value problem:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}+(u\cdot\nabla)u=-\dfrac{1}{\rho}\nabla p+\nu\Delta u+\beta g\theta, & \\ \operatorname{div} u=0, & x\in\Omega,\ t>0, \\ \dfrac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta=\chi\Delta\theta, & \end{cases}$$

$$(2)\qquad \begin{cases} u(x,t)=0,\ \theta(x,t)=\xi(x,t), & x\in\Gamma_1,\ t>0, \\ u(x,t)=0,\ \dfrac{\partial}{\partial n}\theta(x,t)=\eta(x,t), & x\in\Gamma_2,\ t>0, \end{cases}$$

$$(3)\qquad \begin{cases} u(x,0)=a_0(x), & x\in\Omega, \\ \theta(x,0)=\tau_0(x), & \end{cases}$$

where $u=(u_1,u_2,\cdots,u_n)$ is the fluid velocity, p is the pressure, θ is the temperature, $u\cdot\nabla=\sum_{j=1}^n u_j(\partial/\partial x_j)$, $(\partial\theta/\partial n)$ denotes the outer normal derivative of θ at x to $\partial\Omega$, $g(x,t)$ is the gravitational vector function, and ρ(density), ν(kinematic viscosity), β(coefficient of volume expansion), χ(thermal diffusivity) are positive constants. $\xi(x,t)$ (resp. $\eta(x,t)$) is a function defined on $\Gamma_1\times(0,T)$ (resp. $\Gamma_2\times(0,T)$) and $a_0(x)$ (resp. $\tau_0(x)$) is a vector (resp. scalar) function defined on Ω. This system of equations (1) describes the motion of fluid of heat convection (Boussinesq approximation).

In our previous papers [7, 8], we showed the existence of weak solution of the stationary problem. In this paper, we report the existence of a weak solution of evolutional problem (1), (2), (3) (Theorem 1), its uniqueness and some regularity property (Theorems 2, 3), and the existence of solutions with reproductive property (Theorem 4).

Firstly we define some function spaces. The functions considered in this paper are all real valued. $L^p(\Omega)$ and the Sobolev space $W_p^l(\Omega)$ are defined as usual. We also denote $H^l(\Omega)=W_2^l(\Omega)$. Whether the elements of space are scalar or vector functions is understood from the contexts unless stated explicitly.

The solenoidal function spaces are as follows:

$D_\sigma=\{$vector function $\varphi\in C^\infty(\Omega)\,|\,\operatorname{supp}\varphi\subset\Omega,\ \operatorname{div}\varphi=0$ in $\Omega\}$,

H = completion of D_σ under the $L^2(\Omega)$-norm,

V = completion of D_σ under the $H^1(\Omega)$-norm.

It is well known that $V = H_0^1(\Omega) \cap H$, where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the $H^1(\Omega)$ norm ([9]). The following function spaces are also important:

$D_0 = \{$scalar function $\varphi \in C^\infty(\bar{\Omega}) \mid \varphi \equiv 0$ in a neighborhood of $\Gamma_1\}$,

W = completion of D_0 under the $H^1(\Omega)$-norm.

Assume $\{u, p, \theta\}$ be a classical solution of (1), (2), (3). Let us take the L^2 inner product of $v \in D_\sigma$ and the first equation of (1) (resp. $\tau \in D_0$ and the third equation of (1)). Then, using the integration by parts, we obtain:

$$\begin{cases} \dfrac{d}{dt}(u, v) + ((u \cdot \nabla)u, v) = -\nu(\nabla u, \nabla v) + (\beta g\theta, v), \\ \dfrac{d}{dt}(\theta, \tau) + ((u \cdot \nabla)\theta, \tau) = -\chi(\nabla\theta, \nabla\tau) + \chi(\eta, \tau)_{\Gamma_2}, \end{cases} \tag{4}$$

where

$$(\eta, \tau)_{\Gamma_2} = \int_{\Gamma_2} \eta(x')\tau(x')\,d\sigma.$$

Definition 1. A pair of functions $\{u, \theta\}$ is called a weak solution of (1), (2) if $u \in L^2(0, T : V)$, $\theta - \theta_0 \in L^2(0, T : W)$, for some function $\theta_0(x, t)$ in $L^2(0, T : H^1(\Omega))$ such that $\theta_0(x, t) = \xi(x, t)$, $x \in \Gamma_1$, $t \in (0, T)$, and $\{u, \theta\}$ satisfy (4) for any $v \in V$, $\tau \in W$, where the derivative with respect to t is in the distribution sense $\mathcal{D}'(0, T)$.

If we suppose merely $u \in L^2(0, T : V)$ and $\theta - \theta_0 \in L^2(0, T : W)$, the condition (3) doesn't necessarily make sense but according to the following lemma, the condition (3) makes sense.

Lemma 1. *Suppose*

$$g \in L^\infty(\Omega \times (0, T)),\ \theta_0 \in L^2(0, T : H^1(\Omega)),\ \eta \in L^2(\Gamma_2 \times (0, T))$$
$$u \in L^2(0, T : V),\ \theta - \theta_0 \in L^2(0, T : W)$$

and $\{u, \theta\}$ *satisfy* (4) *for any* $v \in V$, $\tau \in W$. *Then* u (*resp.* θ) *is equal to an absolutely continuous function from* $[0, T]$ *into* V' (*resp.* W'), *where* V' (*resp.* W') *is the dual space of* V (*resp.* W).

Our results are the following theorems.

Theorem 1. *Let n be an integer $2 \leq n \leq 4$, and Ω a bounded domain in R^n with C^1 boundary satisfying Condition (H). If $g(x, t)$ is in $L^\infty(\Omega \times (0, T))$, $\xi \in C^1(\bar{\Gamma}_1 \times [0, T])$, $\eta \in L^2(\Gamma_2 \times (0, T))$, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$, then there exists a weak solution $\{u, \theta\}$ of* (1), (2) *satisfying the initial condition* (3). *Furthermore*

$$u \in L^\infty(0, T : H), \quad \theta \in L^\infty(0, T : L^2(\Omega)).$$

Theorem 2. *Let $n = 2$. The weak solution $\{u, \theta\}$ of* (1), (2) *satisfying the initial condition* (3) *is unique. Moreover, u (resp. θ) is almost everywhere equal to a function continuous from $[0, T]$ to H (resp. $L^2(\Omega)$).*

Theorem 3. *Let $n \geq 3$. The weak solution $\{u, \theta\}$ of* (1), (2) *satisfying the initial condition* (3) *is unique if*

$$u \in L^2(0, T : V) \cap L^\infty(0, T : H)$$

$$\theta \in L^2(0, T : H^1(\Omega)) \cap L^\infty(0, T : L^2(\Omega))$$

and

$$u \in L^s(0, T : L^r(\Omega)) \quad \text{and} \quad \theta \in L^s(0, T : L^r(\Omega)),$$

hold for some $r > n$, $s = 2r/(r-n)$.

The proof is based on the construction of approximate solutions by the Galerkin method and a passage to the limit using an a priori estimate on the fractional derivative in time of the approximate solutions and a compactness theorem (cf. J. Leray [3, 4], E. Hopf [1], J. L. Lions [5, 6], R. Temam [9]).

Let $\{u, \theta\}$ be a weak solution of (1), (2). If they satisfy the following condition:

$$u(x, 0) = u(x, T), \quad \theta(x, 0) = \theta(x, T), \tag{5}$$

then we say they have reproductive property (Kaniel-Shinbrot [2]).

Theorem 4. *Let* $2 \leq n \leq 4$, *and* Ω *be a bounded domain in* R^n *with* C^1 *boundary satisfying Condition* (*H*). *Let* $g(x, t)$ *be in* $L^\infty(\Omega \times (0, T))$, $\xi \in C^1(\bar{\Gamma}_1 \times [0, T])$ *and* $\eta \in L^2(\Gamma_2 \times (0, T))$. *Set* $g_\infty = \|g\|_{L^\infty(\Omega \times (0,T))}$. *If* $\beta g_\infty / \sqrt{\nu \chi}$ *is sufficiently small, then there exists a weak solution of* (1), (2) *having reproductive property* (5). *Furthermore*

$$u \in L^\infty(0, T : H), \quad \theta \in L^\infty(0, T : L^2(\Omega)).$$

Using Galerkin's method and Brouwer's theorem, we show the existence of approximate solutions with reproductive property. Its convergence to the weak solution with reproductive property is derived in a similar way to the evolutional case. Full details of the proof will appear elsewhere.

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