82. Some Aspects in the Theory of Representations of Discrete Groups. I[*)]

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Recently representations of infinite discrete groups are studied by many mathematicians. Here we present some new aspects in this field. First we give general criterions for irreducibility and mutual equivalence for a family of induced representations. Then we apply them to infinite wreath product groups. Further we study in detail induced representations of such groups which are far out of reach of the above criterions. These results will be applied to the infinite symmetric group $\mathfrak{S}_\infty$ to get a big family of completely new irreducible unitary representations (=IURs).

1. Induced vectors. Let G be a discrete group and H a subgroup of G. Take a unitary representation π of H on a space $V(\pi)$ and put $U_\pi=\mathrm{Ind}_H^G\pi$. The space $\mathcal{H}(U_\pi)$ for U_π consists of $V(\pi)$-valued functions on G satisfying $f(hg)=\pi(h)f(g)$ $(h\in H,\ g\in G)$, and $\|f\|^2=\sum_{g\in H\backslash G}\|f(g)\|^2<\infty$, where the summation runs over a sections of $H\backslash G$ in G. The representation is given by right translations.

For a vector $v\in V(\pi)$, define an $f\in\mathcal{H}(U_\pi)$ such that $f(e)=v$ and $f(g)=0$ outside of H, where e denotes the unit of G. This f is called the *induced vector* of v and is denoted by $\mathrm{Ind}_H^G v$. The set of all induced vectors in $\mathcal{H}(U_\pi)$ is cyclic in the sense that the G-invariant subspace generated by them is everywhere dense. The notion of induced vectors will be necessary to state equivalence relations among IURs constructed here.

2. Boundedness conditions. Let H_1, H_2 be two subgroups of G, and π_i a unitary representation (=UR) of H_i for $i=1,2$. Put $U_{\pi_i}=\mathrm{Ind}_{H_i}^G\pi_i$, and let $\mathrm{Hom}_G(U_{\pi_1},U_{\pi_2})=\mathrm{Hom}(U_{\pi_1},U_{\pi_2};G)$ be the space of intertwining operators of U_{π_1} with U_{π_2}. Then, every $T\in\mathrm{Hom}_G(U_{\pi_1},U_{\pi_2})$ is given by a kernel $K(g_2,g_1)$, $g_2,g_1\in G$, with values in $\boldsymbol{B}(V(\pi_1),V(\pi_2))$, the space of bounded linear operators of $V(\pi_1)$ into $V(\pi_2)$, as $(Tf)(g)=\sum_{g'\in H_1\backslash G}K(g,g')f(g')$ $(f\in\mathcal{H}(U_{\pi_1}))$, and the kernel satisfies several conditions (cf. [2]).

For $x\in G$, we put ${}^xg=xgx^{-1}$ $(g\in G)$, $H_2^x=x^{-1}H_2x$ and $(\pi_2^x)(h)=\pi_2({}^xh)$ $(h\in H_2^x)$. Then $L=K(x,e)$ belongs to $\mathrm{Hom}(\pi_1,\pi_2^x;H_1\cap H_2^x)$, and it determines $K(g_2,g_1)$ for $g_2g_1^{-1}\in H_2xH_1$. Furthermore we have for L the following two conditions called the *boundedness conditions*: for a positive constant M,

(B_x) $\sum_{h_1}\|L\pi_1(h_1)v\|^2\leqq M\|v\|^2$ $(h_1\in(H_1\cap x^{-1}H_2x)\backslash H_1)$ for $v\in V(\pi_1)$,

(C_x) $\sum_{h_2}\|L^*\pi_2(h_2)w\|^2\leqq M\|w\|^2$ $(h_2\in(H_2\cap xH_1x^{-1})\backslash H_2)$ for $w\in V(\pi_2)$.

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Conversely, for $x \in G$, let d_x be the dimension of the space of $L \in \mathrm{Hom}(\pi_1, \pi_2^x; H_1 \cap H_2^x)$ satisfying (B_x) and (C_x). Then, for a complete system X of representatives of $H_2 \backslash G / H_1$, we have

(1) $$\dim \mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2}) = \sum_{x \in X} d_x.$$

3. Criterions for irreducibility and equivalence. Let $\mathfrak{A}$ be a family of subgroups of G. Consider the following two conditions on $\mathfrak{A}$.

(GRP1) Let $H \in \mathfrak{A}$ and $g \in G$. If $[H : H \cap H^g] < \infty$, then $g \in H$.

(GRP2) Let $H_1, H_2 \in \mathfrak{A}$, and $g \in G$. If $[H_1 : H_1 \cap H_2^g] < \infty$ and $[H_2^g : H_1 \cap H_2^g] < \infty$, then $H_1 = H_2^g$.

Let $\mathfrak{R}$ be a family of IURs of groups in $\mathfrak{A}$. For a pair $\{\pi_1, \pi_2\}$ of elements in $\mathfrak{R}$, we consider the following condition.

(REP) Let π_i be an IUR of $H_i \in \mathfrak{A}$ for $i=1,2$. Suppose, for an $x \in G$, $L \in \mathrm{Hom}(\pi_1, \pi_2^x; H_1 \cap H_2^x)$ satisfies (B_x) and (C_x). Then $L=0$ unless

(2) $$[H_1 : H_1 \cap x^{-1} H_2 x] < \infty \quad \text{and} \quad [H_2 : x H_1 x^{-1} \cap H_2] < \infty.$$

We say that (REP) holds for $\mathfrak{R}$ if it holds for any pair $\pi_1, \pi_2 \in \mathfrak{R}$.

Theorem 1. (i) *Assume that $\mathfrak{A}$ satisfies (GRP1). Let $\pi \in \mathfrak{R}$ be an IUR of $H \in \mathfrak{A}$. If the condition (REP) holds for $\pi_1 = \pi_2 = \pi$, then the induced representation $\mathrm{Ind}_H^G \pi$ is irreducible.*

(ii) *Assume that $\mathfrak{A}$ satisfies (GRP1)–(GRP2). Let $\pi_i \in \mathfrak{R}$ be an IUR of $H_i \in \mathfrak{A}$ for $i=1,2$. If the condition (REP) holds for any pairing $\{\pi_i, \pi_j\}$ $(i, j=1,2)$, then $\mathrm{Ind}_{H_i}^G \pi_i$ are irreducible, and they are mutually equivalent if and only if, for an $x \in G$, $H_1 = H_2^x$, and $\pi_1 \cong \pi_2^x$.*

(iii) *Assume that (GRP1)–(GRP2) hold for $\mathfrak{A}$ and (REP) holds for $\mathfrak{R}$. Then the induced representations of $\pi \in \mathfrak{R}$ are all irreducible, and the conclusion in* (ii) *holds for any pair $\pi_1, \pi_2 \in \mathfrak{R}$.*

We call *elementary* the equivalence relation $U_{\pi_1} \cong U_{\pi_2}$ coming from $\mathrm{Int}(G)$ as in (ii). Let X_f be the subset of X consisting of x for which (2) holds. Then, under the condition (REP), we have

$$\dim \mathrm{Hog}_G(U_{\pi_1}, U_{\pi_2}) = \sum_{x \in X_f} \dim \mathrm{Hom}(\pi_1, \pi_2^x; H_1 \cap H_2^x).$$

We know in [1] that for any pair of finite-dimensional IURs π_i of any subgroups H_i $(i=1,2)$, there holds the condition (REP). Thus

Corollary 2. *Assume (GRP1)–(GRP2) hold for $\mathfrak{A}$, and $\mathfrak{R}$ consists of finite-dimensional IURs. Then the condition (REP) holds for $\mathfrak{R}$. Hence the induced representations U_π, $\pi \in \mathfrak{R}$, are all irreducible, and the equivalence relations among them are all elementary.*

Remark 3. For $G = \mathfrak{S}_\infty$, the infinite symmetric group, Obata's case in [4] and other known cases can be controled by this corollary. However, in our general case, the situation is not so simple and we have non-elementary equivalence relations essentially.

4. Wreath products. For a set A, we denote by $\mathfrak{S}_A$ the group of all finite permutations on A. A permutation σ is called *finite* if it leaves invariant almost all elements in A. Let G_α $(\alpha \in A)$ be a family of discrete groups with an index set A. Then the restricted direct product $G_A = \prod'_{\alpha \in A} G_\alpha$ is defined as the subgroup of the direct product $\prod_{\alpha \in A} G_\alpha$ consisting of

$g=(g_\alpha)_{\alpha\in A}$ with g_α=the unit of G_α for almost all $\alpha \in A$. Let T be a discrete group and define the *wreath product* $\mathfrak{S}_A(T)$ of T with $\mathfrak{S}_A$ as

(3) $\quad \mathfrak{S}_A(T)=D_A(T)\rtimes\mathfrak{S}_A, \quad D_A(T)=\prod'_{\alpha\in A} T_\alpha \quad \text{with } T_\alpha=T\ (\alpha\in A),$

where the product is given by $\sigma\cdot(t_\alpha)_{\alpha\in A}\cdot\sigma^{-1}=(t'_\alpha)_{\alpha\in A}$ with $t'_\alpha=t_{\sigma^{-1}(\alpha)}$ $(\sigma\in\mathfrak{S}_A,$ $t_\alpha\in T_\alpha)$. An element $\sigma\in\mathfrak{S}_A$ which is imbedded into $\mathfrak{S}_A(T)$, is denoted again by σ.

5. Factorizable representations. We consider G_A as a discrete group. A UR of G_A is called *factorizable* if it is equivalent to a direct product $\otimes^a_{\alpha\in A}\pi_\alpha$ of URs π_α of G_α with respect to a reference vector $a=(a_\alpha)_{\alpha\in A}$. Here $\pi^a=\otimes^a_{\alpha\in A}\pi_\alpha$ is defined as follows. The representation space $V(\pi^a)$ is the tensor product $\otimes^a_{\alpha\in A}V_\alpha=\otimes_{\alpha\in A}\{V_\alpha, a_\alpha\}$ of $V_\alpha=V(\pi_\alpha)$ with respect to a reference vector $a=(a_\alpha)_{\alpha\in A}$, $a_\alpha\in V(\pi_\alpha)$, $\|a_\alpha\|=1$, and $\pi^a(g)=\otimes^a_{\alpha\in A}\pi_\alpha(g_\alpha)$ for $g=(g_\alpha)_{\alpha\in A}\in G_A$. For $u=(u_\alpha)_{\alpha\in A}$, $v=(v_\alpha)_{\alpha\in A}$ with unit vectors u_α, $v_\alpha\in V_\alpha$, we consider a relation $\sum_{\alpha\in A}(1-|\langle u_\alpha, v_\alpha\rangle|)<\infty$, and call it the Moore-equivalence and denote it as $u\cong v$.

Let $\otimes^a_{\alpha\in A}\pi_\alpha$ and $\otimes^b_{\alpha\in A}\pi'_\alpha$ be two factorizable representations of G_A, where π_α, π'_α are IURs of G_α $(\alpha\in A)$, $a=(a_\alpha)_{\alpha\in A}$, $a_\alpha\in V(\pi_\alpha)$, $\|a_\alpha\|=1$ and $b=(b_\alpha)_{\alpha\in A}$, $b_\alpha\in V(\pi'_\alpha)$, $\|b_\alpha\|=1$. Then they are mutually equivalent if and only if $\pi_\alpha\cong\pi'_\alpha$ for every $\alpha\in A$, and furthermore $a\cong(K_\alpha b_\alpha)_{\alpha\in A}$, where K_α is a unitary intertwining operator of π'_α with π_α (cf. [3]). We say for this relation that a and b are *Moore-equivalent in an extended sense*.

6. IURs of a wreath product group $\mathfrak{S}_A(T)$. Let T be a finite group. We consider IURs of the wreath product $\mathfrak{S}_A(T)=D_A(T)\rtimes\mathfrak{S}_A$, which come from factorizable URs of $D_A(T)$ or of its subgroups. For a UR π of $D_A(T)$ $=\prod'_{\alpha\in A}T_\alpha$ with $T_\alpha=T$ $(\alpha\in A)$, we put $({}^\sigma\pi)(t)=\pi(\sigma^{-1}t\sigma)$ $(t\in D_A(T))$ and $\mathfrak{S}_A(\pi)$ $=\{\sigma\in\mathfrak{S}_A;\ {}^\sigma\pi\cong\pi\}$ the stationary subgroup of π. Take a UR ρ_T of T and consider a factorizable UR $\pi^a=\otimes^a_{\alpha\in A}\pi_\alpha$ with $\pi_\alpha=\rho_T$ $(\alpha\in A)$ with respect to a reference vector $a=(a_\alpha)_{\alpha\in A}$, $a_\alpha\in V(\pi_\alpha)=V(\rho_T)$, $\|a_\alpha\|=1$. Then, for $\pi=\pi^a$, we have $\mathfrak{S}_A(\pi)=\mathfrak{S}_A$. More exactly, an intertwining operator I_σ of ${}^\sigma\pi$ with π is given as follows. For a decomposable vector $v=\otimes_{\alpha\in A}v_\alpha$ in $V(\pi)=\otimes^a_{\alpha\in A}V_\alpha$, $v_\alpha\in V_\alpha=V(\rho_T)$, put $I_\sigma v=\otimes_{\alpha\in A}v'_\alpha$ with $v'_\alpha=v_{\sigma^{-1}(\alpha)}$ $(\alpha\in A)$. Then $I_{\sigma\sigma'}=I_\sigma I_{\sigma'}$ $(\sigma,\sigma'\in\mathfrak{S}_A)$ and $I_\sigma\circ({}^\sigma\pi)(t)=\pi(t)\circ I_\sigma$ $(t\in D_A(T),\ \sigma\in\mathfrak{S}_A=\mathfrak{S}_A(\pi))$. Take furthermore a character χ of $\mathfrak{S}_A$ and put

(4) $\quad \Pi(t\cdot\sigma)=(\pi(t)I_\sigma)\cdot\chi(\sigma) \qquad (t\in D_A(T),\ \sigma\in\mathfrak{S}_A).$

Then Π gives a UR of $\mathfrak{S}_A(T)$, which is determined by the datum $Q=\{A, \rho_T, \chi, a=(a_\alpha)_{\alpha\in A}\}$ and is also denoted by $\Pi(Q)$. If ρ_T is irreducible, then so is $\Pi(Q)$. We call $\Pi(Q)$ a WP-induced representation of ρ_T.

Generalizing this process, we obtain IURs of $\mathfrak{S}_A(T)$, called *standard.* Such a one $\rho(Q)$ is determined by a datum

(5) $\quad Q=\{(A_\gamma, \rho^\gamma_{T_\gamma}, \chi_\gamma)_{\gamma\in\Gamma}, (a(\gamma))_{\gamma\in\Gamma}, (b_\gamma)_{\gamma\in\Gamma}\},$

where $(A_\gamma)_{\gamma\in\Gamma}$ is a partition of A, T_γ a subgroup of T, $\rho^\gamma_{T_\gamma}$ an IUR of T_γ, χ_γ a character $\mathfrak{S}_{A_\gamma}$, and

$$a(\gamma)=(a_\alpha)_{\alpha\in A_\gamma},\quad a_\alpha\in V_\alpha=V(\rho^\gamma_{T_\gamma}),\quad \|a_\alpha\|=1\ (\alpha\in A_\gamma);$$
$$b_\gamma\in\otimes^{a(\gamma)}_{\alpha\in A_\gamma}V_\alpha,\qquad \|b_\gamma\|=1\ (\gamma\in\Gamma).$$

To give $\rho(Q)$, first take IURs $\Pi(Q_\gamma)$ of $\mathfrak{S}_{A_\gamma}(T_\gamma)$ with data $Q_\gamma=\{A_\gamma, \rho^\gamma_{T_\gamma}, \chi_\gamma, a(\gamma)\}$. Then, consider a subgroup of $\mathfrak{S}_A(T)$ given as $H=H(Q)\equiv \prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}(T_\gamma)$ and an IUR of H through a tensor product: $\pi(Q)=\otimes^b_{\gamma\in\Gamma}\Pi(Q_\gamma)$ with respect to $b=(b_\gamma)_{\gamma\in\Gamma}$, and finally induce it up from H to $\mathfrak{S}_A(T)$: $\rho(Q)=\mathrm{Ind}(\pi(Q)\,;\,H\uparrow\mathfrak{S}_A(T))$.

Putting $\Gamma_f=\{\gamma\in\Gamma\,;\,|A_\gamma|<\infty\}$, $\Gamma_\infty=\{\gamma\in\Gamma\,;\,|A_\gamma|=\infty\}$, with $|A_\gamma|$ the cardinal number of A_γ, we introduce a condition to simplify our statements:

Condition (Q1): $|\Gamma_f|=|\Gamma\backslash\Gamma_\infty|\leqq 1$.

Theorem 4. *For $G=\mathfrak{S}_A(T)$, let $\mathfrak{A}_1$ be the set of subgroups of G of the form $H(Q)$ for which (Q1) holds and $T_\gamma=T$ for $\gamma\in\Gamma_f$. Then the conditions (GRP1)–(GRP2) hold for $\mathfrak{A}_1$. Furthermore the condition (REP) holds for the set $\mathfrak{R}_1$ of all finite-dimensional IURs of $H\in\mathfrak{A}_1$. So Corollary 2 is applicable to $\mathfrak{A}_1$ and $\mathfrak{R}_1$.*

Theorem 5. *Assume A to be countably infinite and let T be a finite group and $\mathfrak{S}_A(T)$ the wreath product group. Then the induced representation $\rho(Q)=\mathrm{Ind}(\pi(Q)\,;\,H(Q)\uparrow\mathfrak{S}_A(T))$ of $\mathfrak{S}_A(T)$ is irreducible if Q in (5) satisfies the condition (Q1) and $\mathrm{Ind}^T_{T_\gamma}\rho^\gamma_{T_\gamma}$ is irreducible for $\gamma\in\Gamma_f$.*

The mutual equivalences among these IURs will be discussed in [5]. Note that they can be described by Theorem 4 for the subfamily of $\rho(Q)$ with $\dim\pi(Q)<\infty$.

References

[1] A. Kleppner: Proc. Amer. Math. Soc., **52**, 731–733 (1961).
[2] G. W. Mackey: Amer. J. Math., **73**, 576–592 (1951).
[3] C. Moore: Ann. Math., **82**, 146–182 (1965).
[4] N. Obata: Nagoya Math. J., **105**, 109–119 (1987); **106**, 143–162 (1987).
[5] T. Hirai: Proc. Japan Acad., **66A**, 16–18 (1990).**)

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