71. On the Unitary Part of Dominant Contractions

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It is known that, for a contraction T (i.e., $\|T\|\leqq 1$) on a Hilbert space H,

$$H_T^{(u)}=\{x\in H\,;\ \|T^n x\|=\|T^{*n}x\|=\|x\|,\ n=1,2,\cdots\}$$
$$=\bigcap_{n=1}^{\infty}\{x\in H\,;\ T^{*n}T^n x=T^n T^{*n}x=x\}$$

is the maximal reducing subspace for T on which the restriction $T|H_T^{(u)}$ of T is unitary. Then we say that $T|H_T^{(u)}$ is the unitary part of T.

A bounded linear operator T on a Hilbert space H is dominant if $(T-\lambda I)H\subset(T-\lambda I)^*H$ for all $\lambda\in\sigma(T)$. It is easily seen, by [1], that this condition is equivalent to the existence of a constant M_λ for each $\lambda\in\mathfrak{C}$ such that $\|(T-\lambda I)^*x\|\leqq M_\lambda\|(T-\lambda I)x\|$ for all $x\in H$. Clearly every hyponormal operator (i.e., $\|Tx\|\geqq\|T^*x\|$ for all $x\in H$) is dominant.

The purpose of this note is to give more precise characterization of $H_T^{(u)}$ for the case where T is a dominant contraction.

Let T be a contraction on H, then the sequence $\{T^{*n}T^n\}$ is monotonically decreasing and hence converges to a non-negative contraction A_T^2 and $T^*A_T^2T=A_T^2$, where A_T is the (unique) non-negative square root of A_T^2. Then we have

Lemma 1. *For any positive integer n,*

$$\|A_T T^n x\|=\|A_T x\|\geqq\|T^{*n}A_T x\| \qquad \textit{for all } x\in H$$

and hence $A_T T^n$ is hyponormal.

Proof. $\|A_T T^n x\|^2=\langle T^{*n}A_T^2T^n x, x\rangle=\langle A_T^2 x, x\rangle=\|A_T x\|^2\geqq\|T^{*n}A_T x\|^2$ for all $x\in H$ because $\|T^*\|\leqq 1$.

Lemma 2. *If $A_T T$ is normal, then A_T is a projection which commutes with T and $H_T^{(u)}=A_T H$.*

Proof. $A_T T$ is normal $\rightleftarrows$ $\|T^*A_T x\|=\|A_T x\|$ for all $x\in H$ by Lemma 1 $\rightleftarrows$ $TT^*A_T x=A_T x$ for all $x\in H$ because $\|T^*\|\leqq 1$ i.e., $TT^*A_T=A_T$.

Then $TA_T^2=T(T^*A_T^2T)=(TT^*A_T)A_T T=A_T^2 T$ and hence A_T commutes with T. Thus $T^{*n}T^nA_T^2=T^{*n}A_T^2T^n=A_T^2$ and $A_T^4=A_T^2$ and hence A_T is a projection.

Next, if $x\in H_T^{(u)}$, then $x=A_T^2x\in A_T H$ and hence $H_T^{(u)}\subset A_T H$. Conversely, for any $x\in H$ and $n=1,2,\cdots$,

$$\|T^n A_T x\|=\|A_T T^n x\|=\|A_T x\|$$

by Lemma 1 and

$$\begin{aligned}\|T^{*n}A_T x\| &= \|T^*A_T(T^{*n-1}x)\| = \|A_T(T^{*n-1}x)\| \\ &= \|T^*A_T(T^{*n-2}x)\| = \|A_T(T^{*n-2}x)\| \\ &= \cdots = \|A_T x\|.\end{aligned}$$

Therefore $A_T H \subset H_T^{(u)}$.

Theorem. *If T^* is a dominant contraction on H, then A_T is a projection which commutes with T and $H_T^{(u)} = A_T H$.*

Proof. By Lemma 2, we have only to prove that $A_T T$ is normal. Let $A_T T = V_T A_T$ is the polar decomposition of $A_T T$. Then $\widetilde{A_T H}$ is invariant under both V_T and T^*. By the decomposition $H = \widetilde{A_T H} \oplus (\widetilde{A_T H})^\perp$, we have

$$V_T = \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}, \quad A_T = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

$V_{11} = V_T \mid \widetilde{A_T H}$ is an isometry because

$$A_T T = V_T A_T = \begin{bmatrix} V_{11}A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

is hyponormal and $0 \notin \sigma_p(V_{11}A_{11})$ by Lemma 1. And it is easily seen that $T_{11}^* = T^* \mid \widetilde{A_T H}$ is also dominant by the definition and that A_{11} is injective with dense range. We obtain

$$A_{11}V_{11}^* = T_{11}^* A_{11} \quad \text{from} \quad A_T T = V_T A_T.$$

Hence V_{11} and T_{11} are normal by [3; Theorem 1] and V_{11} commutes with A_{11} by [2; Theorem 1]. Therefore

$$A_T T = \begin{bmatrix} V_{11}A_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

is normal.

References

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[3] ——: On dominant operators. Monatsh. für Math., **84**, 143–153 (1977).