69. The Plancherel Formula for the Symmetric Space G_C/G_R

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Let G_C be a complex linear connected reductive group, and τ an involutive automorphism of G_C. Denote by G_C^τ the set of all fixed points of τ in G_C, and by $(G_C^\tau)_0$ its identity components. Take a subgroup G_R such that $(G_C^\tau)_0 \subset G_R \subset G_C^\tau$, then $[G_R : (G_R)_0] < \infty$. Let θ be an involution of G_C satisfying that $\theta\tau = \tau\theta$ and $\theta(G) = G$ for $G = G_R$.

Let $\mathfrak{g}_C$ be the complex reductive Lie algebra corresponding to G_C. The automorphisms of $\mathfrak{g}_C$ induced by τ and θ of G_C are denoted by the same letters τ and θ respectively. The decomposition according to the involution τ (resp. θ) are denoted as $\mathfrak{g}_C = \mathfrak{g} + \mathfrak{q}$ (resp. $\mathfrak{g}_C = \mathfrak{k} + \mathfrak{p}$). Let G_H be a complexification of G_C and let $\mathfrak{g}_H$ be its Lie algebra. The dual of $\mathfrak{g}_C$ in $\mathfrak{g}_H$ is defined by $\mathfrak{g}_C^d = \mathfrak{k} \cap \mathfrak{g} + i(\mathfrak{k} \cap \mathfrak{g}) + i(\mathfrak{p} \cap \mathfrak{q}) + \mathfrak{p} \cap \mathfrak{q}$ and the dual of $\mathfrak{k}$ is given by $\mathfrak{k}^d = \mathfrak{k} \cap \mathfrak{g} + i(\mathfrak{p} \cap \mathfrak{q})$ $(i = \sqrt{-1})$. Let K be the analytic subgroup of G_C corresponding to $\mathfrak{k}$, and K^d and G_C^d be those of G_H according to $\mathfrak{k}^d$ and $\mathfrak{g}_C^d$ respectively. In this paper, we study harmonic analysis on the symmetric space $X = G_C/G_R$. The symmetric space G_C/G_R is substantially, "the dual" space of the space $G_R \cong (G_R \times G_R)/G_R$, and we call it the c-dual of the latter. Actually, there exist several dualities between continuous series of X and discrete series of G_R. In § 1, we study continuous series and corresponding invariant spherical distributions on X. In § 2, we discuss general principal series containing the discrete series. In § 3, we study Eisenstein integral and its constant term, and in § 4 Plancherel formula.

§ 1. Continuous series. Here in § 1, we suppose that the symmetric pair $(\mathfrak{g}_C, \mathfrak{g})$ has a sprit Cartan subspace $\mathfrak{a}$. Let $P = MAN$ ($MA = Z_{G_C}(\mathfrak{a})$, $A = \exp \mathfrak{a}$) be the minimal parabolic subgroup associated to $\mathfrak{a}$. Then GP is an open orbit of $G \backslash G_C$. Denote the set of positive roots of $\mathfrak{a}$ associated to P by $\Sigma^+(\mathfrak{a})$, and put $\rho = (1/2)\sum_{\alpha \in \Sigma^+(\mathfrak{a})} \alpha$. Let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$ and $\mathcal{F}$ be a Weyl chamber in $i\mathfrak{a}^*$. If $\alpha \in \Sigma^+(\mathfrak{a})$, let $H_\alpha \in \mathfrak{a}$ be determined by using Killing form: $\langle H_\alpha, H \rangle = \alpha(H)$, $H \in \mathfrak{a}$. We define Poisson kernels $P_\nu(g)$ for $\nu \in \mathcal{F}$ as follows: $P_\nu(g) = \exp \nu\{H(g)\}$ $(g^{-1} \in GM \cdot \exp H(g) \cdot N)$ and $P_\nu(g) = 0$ $(g^{-1} \notin GP)$. Giving a Haar measure dg on G and a G_C-invariant measure dx $(x = gG)$ on X. We define an invariant spherical distribution on X by

$$\Phi_\lambda(f) = \int_X \phi_\lambda(x) f(x)\, dx \qquad (f(x) \in C_C^\infty(X)),$$

where $\phi_\lambda(x) = \int_G P_{\rho-\lambda}(hx)\, dh$. Let W be the Weyl group of the pair $(\mathfrak{g}_C, \mathfrak{a}_C)$ and let W_G be the group defined by $W_G = N_G(A)/Z_G(A)$. We put $W^* = W/W_G$ and define invariant spherical distributions $\Theta_\lambda = \sum_{w \in W^*} \Phi_{w\lambda}$ by taking sum over W^*. Let X' (resp. $\mathcal{F}'$) be the set of all regular elements of X (resp. $\mathcal{F}$). If we normalize the Haar measure on G, then we have:

Proposition 1.1. *The distributions Θ_λ agree with analytic functions on $A'=A\cap X'$. Moreover for $\lambda\in\mathcal{F}'$, we have*

$$\Theta_\lambda|_{A'}=\frac{\sum_{w\in W}\varepsilon(w)e^{w\lambda(X)}}{\pi(\lambda)\Delta(\exp X)}\qquad (X\in\mathfrak{a}),$$

where $\pi(\mu)=\prod_{\alpha>0}\mu(H_\alpha)$.

For a function $f(x)$ on X, we choose an $f_0\in C_C^\infty(G_C)$ such that the integral of f_0 over gG agree with the value $f(x)$ for $x=gG$. For a function $h(g)$ on G_C, we put $h^*(g)=\operatorname{conj} h(g^{-1})$.

Proposition 1.2. *For $f(x)\in C_C^\infty(X)$ satisfying $\operatorname{supp} f\subset G[A]$, we have*

$$\int_{\mathcal{F}'}\Theta_\lambda(f_0^* * f_0)\,|\pi(\lambda)|^2 d\lambda=\int_X |f(x)|^2 dx.$$

Under the assumption of this section, there exist several different types of discrete series on G_R. From the above equality, we may say that the number of the types of discrete series for G_R accords with the multiplicity of the continuous series for G_C/G_R.

§2. Principal series of G_C/G_R. Let $\Pi=\{\mathfrak{j}_1,\mathfrak{j}_2,\cdots,\mathfrak{j}_m\}$ be a maximal set of Cartan subspaces of $\mathfrak{q}$ not conjugate each other under $K\cap G$. For each $\mathfrak{j}_l$ $(l=1,2,\cdots,m)$, we put $\mathfrak{b}_l=\mathfrak{j}_l\cap\mathfrak{k}$ and $\mathfrak{a}_l=\mathfrak{j}_l\cap\mathfrak{p}$. Let $\mathfrak{l}_l=Z_{\mathfrak{g}_C}(\mathfrak{b}_l)$ and take $\mathfrak{m}_l$ such that $\mathfrak{l}_l=\mathfrak{m}_l\oplus\mathfrak{j}_l$. Let $\Sigma^+(\mathfrak{b}_l)$ be the set of all positive roots of $(\mathfrak{b}_l,\mathfrak{g}_C)$, and let $\mathfrak{n}_l=\sum_{\alpha>0}\mathfrak{g}(\mathfrak{b}_l,\alpha)$. The analytic subgroups of G_C according to $\mathfrak{l}_l$, $\mathfrak{m}_l$, $\mathfrak{j}_l$, $\mathfrak{b}_l$ and $\mathfrak{n}_l$ are denoted by L_l, M_l, J_l, B_l and N_l respectively. Then L_l is a linear complex connected reductive group and $P_l=M_lB_lN_l$ is a parabolic subgroup of G_C. If GP_l is a closed (resp. open) orbit in $G\backslash G_C$, it corresponds to the discrete (resp. continuous) series. The symmetric space $M_l/M_l\cap G$ has continuous series. Let W^* be the Weyl group of the pair $(\mathfrak{a}_l,\mathfrak{m}_l)$. We choose a Weyl chamber $\mathcal{F}_l$ in $i\mathfrak{a}_l^*$. For $\lambda\in\mathcal{F}_l'$, let Θ_λ^l be the invariant spherical distributions on $M_l/M_l\cap G$ given in §1. Parametrize the discrete series ω by $(B_l\cap G\backslash B_l)^\wedge$ and define a Poisson kernel P_ω by $P_\omega(g)=\exp[(\rho-\omega)U(g)]$ for $g^{-1}\in Gm(g)\cdot\exp U(g)\cdot N_l$ $(m(g)\in M_l\cap G\backslash M_l$, $\exp U(g)\in B_l\cap G\backslash B_l)$, $P_\omega(g)=0$ for $g^{-1}\notin GP_l$. And we define invariant spherical distributions $\Theta_{\lambda,\omega}^l$ on X by

$$\Theta_{\lambda,\omega}^l(f)=\int_X \theta_{\lambda,\omega}^l(x)f(x)\,dx\qquad (f(x)\in C_C^\infty(X)),$$

where $\theta_{\lambda,\omega}^l(x)=\int_G\Theta_\lambda^l(m(hx))\,P_{\rho-\omega}(hx)\,dh$. We put $W_G^l=N_G(\mathfrak{j}_l)/Z_G(\mathfrak{j}_l)$ and W_l the Weyl group of the pair $(\mathfrak{j}_l,\mathfrak{g}_C)$. Let W_l^* be the subgroup of W_l generated by W^* and W_G^l. Denote by Π_l the set of all Cartan subspaces in Π obtained from $\mathfrak{j}_l$ through Cayley transformations corresponding to imaginary roots of $\Sigma(\mathfrak{j}_l)$ and conjugations of $K\cap G$.

Proposition 2.1. *For $\lambda\in\mathcal{F}'$ and $\omega\in(B_l\cap G\backslash B_l)^\wedge$, the distributions $\Theta_{\lambda,\omega}^l$ agree on $B_l\cap G\backslash B_l'$ with analytic functions which are given by*

$$\Theta^l_{\lambda,\omega}|_{J_l \cap G \backslash J_{l'}} = \begin{cases} \dfrac{\sum_{w \in W_l^*} \varepsilon(w) e^{w(\lambda,\omega)(X)}}{\pi(\lambda,\omega)\Delta(\exp X)} & X \in \mathfrak{j}_l \\ 0 & X \in \mathfrak{j} \ (\mathfrak{j} \in \Pi \backslash \Pi_l) \end{cases}$$

§ 3. **Eisenstein integral.** We define for the parabolic subgroup $P_l = M_l B_l N_l$, an integral as follows. For $\varphi \in C^\infty(M_l/M_l \cap G)$, extend it to a function on G_C by $\varphi\ (hwnbm) = \varphi(m)$. For $g \in G_C$, let $B(g)$ denote the element of $(B_l \cap G)\backslash B_l$ given as $g \in Gw(g)N_l B(g) M_l$ $(B(g) \in (B_l \cap G)\backslash B_l$, $G_C = \bigcup_{w \in W} GwP_l)$. Then we define

$$E(P_l : \varphi : x) = \int_G \alpha(h)\varphi(xh) \exp(\rho - w)B(xh)\,dh, \text{ for } \alpha \in C_C^\infty(G).$$

We next consider the constant term on symmetric spaces. For $P = M_l B_l N_l$, let $P^d = M_l^d A_l^d N_l^d$ be the dual of P in G_C^d, then $M_l^d \cong M$, $N_l^d \cong N_l$. For the spaces

$$C^\infty(P/G) = \{f \in C^\infty(X) : \operatorname{supp} f \subset P_l/G\},$$
$$C^\infty(P^d/K^d) = C^\infty(G_C^d/K^d),$$

consider the isomorphism

$$\eta : C^\infty(P/G) \overset{\text{local}}{\cong} C^\infty(P^d/K^d) : f \to f^\eta.$$

And denote by η_l the analogous isomorphism in the case of M_l. For $f \in \mathcal{A}(X)$, we define its constant term $f_P = \eta_l^{-1}(f^\eta_{P^d})$ on $M_l/M_l \cap G$ by

$$\lim_{\substack{a \to \infty \\ P^d}} \{d_{P^d}(am) f^\eta(am) - f^\eta_{P^d}(am)\} = 0.$$

Proposition 3.1. *Let P and P' be parabolic subgroups whose compact part is $\mathfrak{b}$. Then there exist constants $c(s,\omega)$ $(s \in W(\mathfrak{b}))$ satisfying*

$$\underset{P'}{E}(P : \varphi : bm) = \sum_{s \in W(\mathfrak{b})} (c(s,\omega)\varphi)(m) e^{s\omega(\log b)}.$$

§ 4. **Plancherel formula.** Here we discuss Plancherel formula by using the invariant spherical distributions $\Theta^l_{\lambda,\omega}$ defined in § 2. The Eisenstein integral corresponding to $\Theta^l_{\lambda,\omega}$ is given as follows:

Proposition 4.1. *For $\lambda \in \mathfrak{F}_l$ and $\omega \in (B_l/B_l \cap G)^\wedge$, we have*

$$\langle \Theta^l_{\lambda,\omega}, l(x)f \rangle = E(P_l : \varphi_f : x)$$

where, with $xh = h_0 w n_0 b_0 m_0$,

$$\varphi_f = \int_{M_l/M_l \cap G} \int_{N_l \times (B_l/B_l \cap G)} \Theta_\lambda(m) \exp\{(\omega - \rho)(\log b)\} f(nbm_0 m)\,dndb^*dm^*.$$

If we normalize the measures, we have the Plancherel formula as follows:

Theorem 4.2. *For $f \in C_C^\infty(X)$, we have*

$$\int_X |f(x)|^2 dx = \sum_{l=1}^m \sum_{\omega \in (B_l/B_l \cap G)^\wedge} \int_{\mathfrak{F}_{l'}} \langle \Theta^l_{\lambda,\omega}, f_0^* * f_0 \rangle |\pi(\lambda,\omega)|^2 d\lambda.$$

Outline of proof. For each $\Theta^l_{\lambda,\omega}$, we reduce it to the function on $M_l/M_l \cap G$ by Proposition 4.1 and by taking its constant term. For the symmetric space $M_l/M_l \cap G$, Proposition 1.2 holds. We decompose the space $C_C^\infty(X)$ according to each parabolic subgroup. We give Fourier inversion formulas associated to each part of the decompositions. Then we get the Plancherel formula after combining these formulas.

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