58. A Weak Convergence Theorem in Sobolev Spaces with Application to Filippov's Evolution Equations

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1. Introduction. Let $\mathfrak{H}$ be a real separable Hilbert space. A correspondence (=multi-valued mapping) $\Gamma: [0, T]\times\mathfrak{H} \twoheadrightarrow \mathfrak{H}$ is assumed to be given. A double arrow $\twoheadrightarrow$ is used in order to indicate the domain and the range of a correspondence. The compact interval $[0, T]$ is endowed with the usual Lebesgue measure dt. The target of this paper is to establish a sufficient condition which assures the existence of solutions of a multi-valued differential equation of the form:

$$(*) \qquad \dot{x}(t) \in \Gamma(t, x(t)), \qquad x(0)=a,$$

where a is a fixed vector in $\mathfrak{H}$.

In Maruyama [8], I have already presented a solution of this problem in the special case of $\mathfrak{H}=\boldsymbol{R}^l$ by making use of the convenient properties of the weak convergence in the Sobolev space $\mathfrak{W}^{1,2}([0, T], \boldsymbol{R}^l)$ consisting of functions of $[0, T]$ into $\boldsymbol{R}^l$; i.e. if a sequence $\{x_n\}$ in $\mathfrak{W}^{1,2}([0, T], \boldsymbol{R}^l)$ weakly converges to some $x^* \in \mathfrak{W}^{1,2}([0, T], \boldsymbol{R}^l)$, then

$$x_n \to x^* \quad \text{strongly in } \mathfrak{L}^1([0, T], \boldsymbol{R}^l), \quad \text{and}$$
$$\dot{x}_n \to \dot{x}^* \quad \text{weakly in } \mathfrak{L}^2([0, T], \boldsymbol{R}^l).$$

However it is well-known that this property does not hold in the space $\mathfrak{W}^{1,2}([0, T], \mathfrak{H})$ consisting of functions of $[0, T]$ into $\mathfrak{H}$ if $\dim \mathfrak{H}=+\infty$. (Cf. Cecconi [5] pp. 28–29.) We shall first provide a new tool to overcome this difficulty in section 2, and then proceed to the existence theorem for the differential equation $(*)$ in section 3.

2. A convergence theorem in $\mathfrak{W}^{1,p}([0, T], \mathfrak{H})$. We denote by $\mathfrak{H}_s$ (resp. $\mathfrak{H}_w$) the Hilbert space $\mathfrak{H}$ endowed with the strong (resp. weak) topology.

Theorem 1. *Let $\mathfrak{H}$ be a real separable Hilbert space and consider a sequence $\{x_n\}$ in the Sobolev space $\mathfrak{W}^{1,p}([0, T], \mathfrak{H})$ $(p\geqq 1)$. Assume that*

(i) *the set $\{x_n(t)\}_{n=1}^{\infty}$ is bounded (and hence relatively compact) in $\mathfrak{H}_w$ for each $t \in [0, T]$, and*

(ii) *there exists some function $\psi \in \mathfrak{L}^p([0, T], (0, +\infty))$ such that*

$$\|\dot{x}_n(t)\| \leqq \psi(t) \quad a.e.$$

Then there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and some $x^* \in \mathfrak{W}^{1,p}([0, T], \mathfrak{H})$ such that

(a) *$z_n \to x^*$ uniformly in $\mathfrak{H}_w$ on $[0, T]$, and*

(b) *$\dot{z}_n \to \dot{x}^*$ weakly in $\mathfrak{L}^p([0, T], \mathfrak{H})$.*

Proof. (a) To start with, we shall show the equicontinuity of $\{x_n\}$. Since ψ is integrable, there exists some $\delta > 0$ for each $\varepsilon > 0$ such that

$$\|x_n(t)-x_n(s)\| \leqq \int_s^t \|\dot{x}_n(\tau)\| d\tau \leqq \int_s^t \psi(\tau)d\tau \leqq \varepsilon \qquad \text{for all } n$$

provided that $|t-s| \leqq \delta$. This proves the equicontinuity of $\{x_n\}$ in the strong topology for $\mathfrak{H}$. Hence $\{x_n\}$ is also equicontinuous in the weak topology for $\mathfrak{H}$.

Taking account of this fact as well as the assumption (i), we can claim, thanks to the Ascoli-Arzelà theorem (cf. Schwartz [12] p. 78), that $\{x_n\}$ is relatively compact in $\mathfrak{C}([0, T], \mathfrak{H}_w)$ (the set of continuous functions of $[0, T]$ into $\mathfrak{H}_w$) with respect to the topology of uniform convergence.

By the assumption (i), $\{x_n(0)\}$ is bounded in $\mathfrak{H}$, say

$$\sup_n \|x_n(0)\| \leqq C < +\infty.$$

And the assumption (ii) implies that

$$\left\| \int_0^t \dot{x}_n(\tau)d\tau \right\| \leqq \|\psi\|_1 \qquad \text{for all } t \in [0, T].$$

Hence

$$\sup_n \|x_n(t)\| = \sup_n \left\| x_n(0) + \int_0^t \dot{x}_n(\tau)d\tau \right\|$$
$$\leqq C + \|\psi\|_1 \qquad \text{for all } t \in [0, T].$$

Thus each x_n can be regarded as a mapping of $[0, T]$ into the set

$$M \equiv \{w \in \mathfrak{H} \mid \|w\| \leqq C + \|\psi\|_1\}.$$

The weak topology on M is metrizable because M is bounded and $\mathfrak{H}$ is a separable Hilbert space. Hence if we denote by M_w the space M endowed with the weak topology, then the uniform convergence topology on $\mathfrak{C}([0, T], M_w)$ is metrizable.

Since we can regard $\{x_n\}$ as a relatively compact subset of $\mathfrak{C}([0, T], M_w)$, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ which uniformly converges to some $x^* \in \mathfrak{C}([0, T], \mathfrak{H}_w)$.

(b) Since

$$\|\dot{y}_n(t)\| \leqq \psi(t) \quad \text{a.e.},$$

the sequence $\{w_n : [0, T] \to \mathfrak{H}\}$ defined by

$$w_n(t) = \frac{\dot{y}_n(t)}{\psi(t)}; \quad n = 1, 2, \cdots$$

is contained in the unit ball of $\mathfrak{L}^\infty([0, T], \mathfrak{H})$ which is weak*-compact by Alaoglu's theorem. Note that the weak*-topology on the unit ball of $\mathfrak{L}^\infty([0, T], \mathfrak{H})$ is metrizable since $\mathfrak{L}^1([0, T], \mathfrak{H})$ is separable. Hence $\{w_n\}$ has a subsequence $\{w_{n'}\}$ which converges to some $w^* \in \mathfrak{L}^\infty([0, T], \mathfrak{H})$ in the weak*-topology. We shall write $\dot{z}_n = \dot{y}_{n'} = \psi \cdot w_{n'}$.

If we define an operator $A : \mathfrak{L}^\infty([0, T], \mathfrak{H}) \to \mathfrak{L}^p([0, T], \mathfrak{H})$ by

$$A : g \longmapsto \psi \cdot g,$$

then A is continuous in the weak*-topology for $\mathfrak{L}^\infty$ and the weak-topology for $\mathfrak{L}^p$. In order to see this, let $\{g_\lambda\}$ be a net in $\mathfrak{L}^\infty([0, T], \mathfrak{H})$ such that $w^*\text{-}\lim_\lambda g_\lambda = g^* \in \mathfrak{L}^\infty([0, T], \mathfrak{H})$; i.e.

$$\int_0^T \langle \alpha(t), g_\lambda(t) \rangle dt \to \int_0^T \langle \alpha(t), g^*(t) \rangle dt \qquad \text{for all } \alpha \in \mathfrak{L}^1([0, T], \mathfrak{H}).$$

Then it is quite easy to verify that

$$\int_0^T \langle \beta(t), \psi(t)g_\lambda(t)\rangle dt = \int_0^T \langle \psi(t)\beta(t), g_\lambda(t)\rangle dt \to \int_0^T \langle \psi(t)\beta(t), g^*(t)\rangle dt$$

for all $\beta \in \mathfrak{L}^q([0, T], \mathfrak{H})$, $1/p+1/q=1$

since $\psi \cdot \beta \in \mathfrak{L}^1([0, T], \mathfrak{H})$. This proves the continuity of A.

Hence

(1) $$\dot{z}_n = \psi \cdot w_{n'} \to \psi \cdot w^* \qquad \text{weakly in } \mathfrak{L}^p([0, T], \mathfrak{H}),$$

which implies

(2) $$\left\langle \theta, \int_s^t \dot{z}_n(\tau)d\tau \right\rangle = \int_s^t \langle \theta, \dot{z}_n(\tau)\rangle d\tau \to \int_s^t \langle \theta, \psi(\tau)\cdot w^*(\tau)\rangle d\tau \qquad \text{for all } \theta \in \mathfrak{H}.$$

On the other hand, since

$$z_n(t) - z_n(s) = \int_s^t \dot{z}_n(\tau)d\tau \qquad \text{for all } n,$$

and $z_n(t)-z_n(s) \to x^*(t)-x^*(s)$ in $\mathfrak{H}_w$, we get

(3) $$\left\langle \theta, \int_s^t \dot{z}_n(\tau)d\tau \right\rangle = \langle \theta, z_n(t)-z_n(s)\rangle \to \langle \theta, x^*(t)-x^*(s)\rangle \qquad \text{for all } \theta \in \mathfrak{H}.$$

(2) and (3) imply that

$$\langle \theta, x^*(t)-x^*(s)\rangle = \left\langle \theta, \int_s^t \psi(\tau)\cdot w^*(\tau)d\tau \right\rangle \qquad \text{for all } \theta \in \mathfrak{H},$$

from which we can deduce the equality

(4) $$x^*(t)-x^*(s) = \int_s^t \psi(\tau)\cdot w^*(\tau)d\tau.$$

By (1) and (4), we get the desired result:

$$\dot{z}_n \to \dot{x}^* = \psi \cdot w^* \qquad \text{weakly in } \mathfrak{L}^p([0, T], \mathfrak{H}).$$ Q.E.D.

In the proof of our Theorem 1, we are making use of some ideas of Aubin and Cellina [4] (pp. 13–14). However their reasoning does not seem to be perfectly sound.

3. Multi-valued differential equation (*). Let us begin by specifying some assumptions imposed on the correspondence $\Gamma: [0, T]\times \mathfrak{H}_w \twoheadrightarrow \mathfrak{H}_s$. Special attentions should be paid to the fact that the topology on the domain is the weak one and the range is endowed with the strong topology.

Assumption 1. *Γ is compact-convex-valued; i.e. $\Gamma(t, x)$ is a non-empty, compact and convex subset of $\mathfrak{H}$ for all $t \in [0, T]$ and all $x \in \mathfrak{H}$.*

Assumption 2. *The correspondence $x \mapsto\!\!\!\twoheadrightarrow \Gamma(t, x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t \in [0, T]$; i.e. for any fixed $(t, x) \in [0, T]\times \mathfrak{H}_w$ and for any neighborhood V of $\Gamma(t, x) \subset \mathfrak{H}_s$, there exists some neighborhood U of x such that $\Gamma(t, z) \subset V$ for all $z \in U$.*

Assumption 3. *The correspondence $t \mapsto\!\!\!\twoheadrightarrow \Gamma(t, x)$ is measurable for each fixed $x \in \mathfrak{H}$; i.e. the weak inverse image $\Gamma^{-w}(U) = \{t \in [0, T] \mid \Gamma(t, x) \cap U \neq \emptyset\}$ is measurable for all open sets U in $\mathfrak{H}_s$ and for each fixed $x \in \mathfrak{H}$.* (For the concept of "measurability" of a correspondence, see Castaing-Valadier [4] Chap. III or Maruyama [9] Chap. 7–8.)

Assumption 4. *There exists $\psi \in \mathfrak{L}^2([0, T], (0, +\infty))$ such that $\Gamma(t, x) \subset S_{\psi(t)}$ for every $(t, x) \in [0, T]\times \mathfrak{H}$, where $S_{\psi(t)}$ is the closed ball in $\mathfrak{H}$ with the center 0 and the radius $\psi(t)$.*

Remark. Among other things, the assumption that the set $\Gamma(t, x)$ is always convex is seriously restrictive, especially from the viewpoint of applications. However there seems to be no easy way to wipe out the convexity assumption. (See De Blasi [6].)

We are now going to find out a solution of (∗) in the Sobolev space $\mathfrak{W}^{1,2}([0, T], \mathfrak{H})$. Define a set $\Delta(a)$ in $\mathfrak{W}^{1,2}$ by

$$\Delta(a)=\{x \in \mathfrak{W}^{1,2} \mid x \text{ satisfies } (*) \text{ a.e.}\}$$

for a fixed $a \in \mathfrak{H}$. The following theorem tells us that $\Delta(a)\neq\emptyset$ and that Δ depends continuously, in some sense, upon the initial value a.

Theorem 2. *Suppose that Γ satisfies Assumptions* 1–4, *and let A be a non-empty, convex and compact subset of $\mathfrak{H}_s$. Then*

(i) *$\Delta(a^*)\neq\emptyset$ for any $a^* \in A$, and*

(ii) *the correspondence $\Delta: A \twoheadrightarrow \mathfrak{W}^{1,2}$ is compact-valued and u.h.c. on A, in the weak topology for $\mathfrak{W}^{1,2}$.*

Outline of Proof. (I) If we define a subset $\mathfrak{X}$ of the Sobolev space $\mathfrak{W}^{1,2}([0, T], \mathfrak{H})$ by

$$\mathfrak{X}=\{x \in \mathfrak{W}^{1,2} \mid \|\dot{x}(t)\|\leqq\psi(t) \text{ a.e. and } x(0) \in A\},$$

then $\mathfrak{X}$ is a non-empty, convex and weakly compact subset of $\mathfrak{W}^{1,2}$. We can also show that the set

$$H=\{(a, x, y) \in A\times\mathfrak{X}\times\mathfrak{X} \mid \dot{y}(t) \in \Gamma(t, x(t)) \text{ a.e. and } x(0)=y(0)=a\}$$

is weakly compact in $A\times\mathfrak{X}\times\mathfrak{X}$. This fact, the proof of which is based upon Theorem 1, provides a crucial key for the proof of Theorem 2.

(II) Fix any $a^* \in A$. If we define a set $\mathfrak{X}'\subset\mathfrak{X}$ by $\mathfrak{X}'=\{x \in \mathfrak{X} \mid x(0)=a^*\}$, then $\mathfrak{X}'$ is convex and weakly compact in $\mathfrak{W}^{1,2}$. Furthermore we define a correspondence $\Phi: \mathfrak{X}' \twoheadrightarrow \mathfrak{X}'$ by

$$\Phi(x)=\{y \in \mathfrak{X}' \mid \dot{y}(t) \in \Gamma(t, x(t)) \text{ a.e.}\}.$$

Then the problem is simply reduced to finding out a fixed point of Φ.

1° $\Phi(x)\neq\emptyset$ for every $x \in \mathfrak{X}'$. ——This fact can be proved through the Measurable Selection Theorem (cf. Castaing-Valadier [3] Chap. III or Maruyama [9] Chap. 7).

2° Φ is convex-compact-valued. ——This is not hard.

3° Φ is u.h.c. ——If we define the a^*-section H_{a^*} of H by $H_{a^*}=\{(a, x, y) \in H \mid a=a^*\}$, then H_{a^*} is obviously weakly compact in $A\times\mathfrak{X}\times\mathfrak{X}$. And the graph $G(\Phi)$ of Φ is expressed as $G(\Phi)=\mathrm{proj}_{\mathfrak{X}\times\mathfrak{X}} H_{a^*}$, the projection of H_{a^*} into $\mathfrak{X}\times\mathfrak{X}$, which is also closed.

Summing up ——Φ is convex-compact-valued and u.h.c. Applying now Ky Fan's Fixed-Point Theorem (Fan [7]) to the correspondence Φ, we obtain an $x^* \in \mathfrak{X}'$ such that $x^* \in \Phi(x^*)$; i.e.

$$\dot{x}^*(t) \in \Gamma(t, x^*(t)) \text{ a.e.} \quad \text{and} \quad x^*(0)=a^*.$$

This proves (i).

(III) Since the compactness of $\Delta(a)$ can be verified by applying Mazur's theorem and making use of Assumptions 1–2, we may omit the details. Hence we have only to show the u.h.c. of Δ. However it is also obvious because the graph $G(\Delta)$ of Δ can be expressed as

$$G(\Delta)=\mathrm{proj}_{A\times\mathfrak{X}}\{(a, x, y)\in H \mid x=y\},$$

which is closed in $A\times\mathfrak{X}$. Q.E.D.

I am much indebted to Castaing-Valadier [3] for various important ideas embodied in the proof of Theorem 2. See also Maruyama [10] for details.

Here it may be suggestive for us to glimpse the special case in which Γ is a (single-valued) mapping.

Corollary 1. *Let* $f: [0, T]\times\mathfrak{H}_w\to\mathfrak{H}_s$ *be a (single-valued) mapping which satisfies the following three conditions.*

(i) *The function* $x\mapsto f(t, x)$ *is continuous for each fixed* $t\in[0, T]$.

(ii) *The function* $t\mapsto f(t, x)$ *is measurable for each fixed* $x\in\mathfrak{H}$.

(iii) *There exists* $\psi\in\mathfrak{L}^2([0, T], (0, +\infty))$ *such that* $f(t, x)\in S_{\psi(t)}$ *for every* $(t, x)\in[0, T]\times\mathfrak{H}$; *i.e.* $\sup_{x\in\mathfrak{H}}\|f(t, x)\|\leqq\psi(t)$ *for all* $t\in[0, T]$.

Then the differential equation

$$(**)\qquad \dot{x}=f(t, x),\quad x(0)=a\quad (\textit{fixed vector in } \mathfrak{H})$$

has at least a solution in $\mathfrak{W}^{1,2}([0, T], \mathfrak{H})$. (A solution of (**) is a function $x\in\mathfrak{W}^{1,2}$ which satisfies (**) a.e.)

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