57. *Convergence Theorems for the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation*

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§ 1. **Introduction.** L^{α} denotes the space of α-summable function on $\boldsymbol{R}^N$ with the norm $\|\cdot\|_{\alpha}$. H^s represents the standard Sobolev space of order s on $\boldsymbol{R}^N$. We will use the abbreviation $\|\cdot\|=\|\cdot\|_2$. We put $i=\sqrt{-1}$, $\sigma=2+4/N$, $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$ $(j=1, \cdots, N)$, $\nabla=(\partial_1, \cdots, \partial_N)$ and $\Delta=\nabla\cdot\nabla$ (Laplace operator on $\boldsymbol{R}^N$). $(I; L^{\alpha})$ denotes the space of continuous functions from a interval $I\subset\boldsymbol{R}$ to L^{α} with the norm $|\cdot|_{\alpha,\infty,I}=\sup_{t\in I}\|\cdot(t)\|_{\alpha}$. If $I=\boldsymbol{R}$, we will use $|\cdot|_{\alpha,\infty,\boldsymbol{R}}=|\cdot|_{\alpha,\infty}$. μ denotes the Lebesgue measure on $\boldsymbol{R}^N$. For brevity we write $[f>\eta]=\{x\in\boldsymbol{R}^N; f(x)>\eta\}$.

This paper is concerned with the following Cauchy problem for the nonlinear Schrödinger equation:

$$C(p)\qquad \begin{array}{l} 2i\partial_t u+\Delta u+|u|^{p-1}u=0, \quad (t, x)\in\boldsymbol{R}\times\boldsymbol{R}^N, \\ u(0, x)=u_0(x) \quad x\in\boldsymbol{R}^N, \end{array}$$

where $1<p<2^*-1$ ($2^*=2N/(N-2)$ if $N\geqq 3$, arbitrary number larger than 2 if $N=1$ and 2). It is well known that for any $u_0\in H^1$, there exist an open interval I in $\boldsymbol{R}$ containing the origin and a unique solution $u_p(t, x)$ of $C(p)$ in $C(I; H^1)$ which satisfies two conservation laws;

$$(1)\qquad \|u_p(t)\|=\|u_0\|,$$

$$(2)\qquad E_{p+1}(u_p)=\|\nabla u_p\|^2-\frac{2}{p+1}\|u_p\|_{p+1}^{p+1}=E_{p+1}(u_0).$$

If $1<p<1+4/N$, u_p exists globally in time, i.e., $I=\boldsymbol{R}$ by (2) and the Gagliardo-Nirenberg inequarity. That is, there is a positive constant $C(p, E_p)$ such that

$$(3)\qquad |\nabla u_p|_{2,\infty}, \qquad |u_p|_{p+1,\infty}<C(p, E_p).$$

If $p\geqq 1+4/N$, however, there exist singular solar solutions exploding their L^2 norms of the gradient in finite time (blow-up): Each singular solution $u(t)$ shows that

$$(4)\qquad \lim_{t\to T}\|\nabla u(t)\|=\infty \qquad \text{for some } T\in\boldsymbol{R}.$$

So it can occur that

$$(5)\qquad \limsup_{p\uparrow 1+4/N}|\nabla u_p|_{2,\infty}=\limsup_{p\uparrow 1+4/N}|u_p|_{\sigma,\infty}=\infty.$$

Thus our purpose is to obtain more precise analysis of the behavior of (u_p) as $p\uparrow 1+4/N$ in $C(\boldsymbol{R}; L^{\sigma})$ (or $C(\boldsymbol{R}; H^1)$). We will consider the rescaling function:

$$(6)\qquad u_{p,\lambda}(t, x)=\lambda_p^{N/2}u_p(\lambda_p^2 t, \lambda_p x),$$

where

$$(7)\qquad \lambda_p=1/|u_p|_{\sigma,\infty}^{\sigma/2} \quad (\to 0 \text{ as } p\uparrow 1+4/N).$$

This leads in a natural way to the consideration of functions $u(t, x)$ in $C(\boldsymbol{R}; H^1)$ satisfying the pseudo-conformally invariant nonlinear Schrödinger equation:

(LP) $$2i\partial_t u + \Delta u + \lambda |u|^{4/N} u = 0, \quad (t, x) \in \boldsymbol{R} \times \boldsymbol{R}^N,$$

where

(8) $$(0 \neq) \quad \lambda \equiv \lim_{P \uparrow 1+4/N} \lambda_P^{-N(P+1-\sigma)/2} \quad (\leqq 1).$$

Here we note that (LP) arises in the nonlinear optics as a model of the self-focusing of a laser beam.

Our main result is the following

Theorem A. *Let* $\{u_P\}$ *be a family of solutions of* $C(p)$*'s for* $1 < p < 1 + 4/N$ *in* $C(\boldsymbol{R}; H^1)$ *with* (5). *Let* $\{p_n\}$ *be a sequence such that* $p_n \uparrow 1 + 4/N$ *and* $\lim_{n\to\infty} |\nabla u_P|_{2,\infty} = \lim_{n\to\infty} |u_P|_{\sigma,\infty} = \infty$ *as* $n \to \infty$. *Set*

(A.1) $$\lambda_n = \lambda_{P_n}, \qquad u_n(t, x) = u_{P_n, \lambda_n}(t, x),$$

(A.2) $$E_{\sigma,\lambda}(v) = \|\nabla v\|^2 - \frac{2}{\sigma} \lambda \|v\|_\sigma^\sigma.$$

Then there exists a subsequence of $\{u_n\}$ *(we still denote it by* $\{u_n\}$*) which satisfies the following properties: one can find* $L \in \boldsymbol{N}$*, solutions* $\{u^j\}$ *of (LP) in* $C(\boldsymbol{R}; H^1)$ *with* $E_{\sigma,\lambda}(u^j) = 0$ *and sequences* $\{(s_n, y_n^j)\}$ *in* $\boldsymbol{R} \times \boldsymbol{R}^N$ *for* $1 \leqq j \leqq L$ *such that*

(A.3) $$\lim_{n\to\infty} |(s_n, y_n^j) - (s_n, y_n^k)| = \infty \quad (j \neq k),$$

(A.4) $$u_n^1 \equiv u_n(\cdot + s_n, \cdot + y_n^1) \to u^1 \text{ weakly* in } L^\infty(\boldsymbol{R}, H^1),$$

(A.5) $$u_n^j \equiv (u_n^{j-1} - u^{j-1})(\cdot, \cdot + y_n^j) \to u^j \quad (j \geqq 2) \quad \text{weakly* in } L^\infty(\boldsymbol{R}; H^1),$$

(A.6) $$\lim_{n\to\infty} \int_I \{E_{\sigma,\lambda}(u_n^j) - E_{\sigma,\lambda}(u_n^j - u^j) - E_{\sigma,\lambda}(u^j)\} dt = 0, \qquad \text{for any } I \Subset \boldsymbol{R},$$

(A.7) $$\lim_{n\to\infty} \|u_n^1(0) - u^1(0)\|_\sigma = 0.$$

Corollary B. *Let* Q *be a nontrivial minimal* L^2 *norm solution to* $\Delta Q - Q + |Q|^{4/N} = 0$ $(Q \in H^1)$. *If* $\|u_P(t)\| = \|Q\|$*, then we have* $L = 1$ *(in Theorem A) and*

(B.1) $$\lim_{n\to\infty} \|u_n^1(0) - u^1(0)\|_2 = \lim_{n\to\infty} \|\nabla u_n^1(0) - \nabla u^1(0)\|_2 = 0,$$

where u^1 *is a solution of (LP) with* $\lambda = 1$.

§ 2. Sketch of proof. First we note that the rescaling funcion u_n is a solution of

(9) $$2i\partial_t u_n + \Delta u_n + \lambda_n^{-N(P+1-\sigma)/2} |u_n|^{P-1} u_n = 0,$$

and satisfies

(10) $$\|u_n\| = \|u_0\|, \qquad |u_n|_{\sigma,\infty} = 1,$$

(11) $$\limsup_{n\to\infty} E_{\sigma,\lambda}(u_n) \leqq 0, \text{ for some } t \in \boldsymbol{R}.$$

Thus one can see that $\{u_n\}$ is a bounded sequence in $L^\infty(\boldsymbol{R}; H^1)$ by (11) and the Gagliardo-Nirenberg, so that we have from (10) and (11),

Lemma 1. u_n *satisfies*

(12) $$\sup_{t\in\boldsymbol{R}} \mu([|u_n(t, \cdot)| > \eta]) > C$$

for some constants η, $C > 0$ *independent of* n.

We proceed.

Lemma 2. $\{u_n\}$ *is an equicontinuous family in* $C(\boldsymbol{R}; L^2)$*, and form an equibounded family in* $C(\boldsymbol{R}; H^1)$ *such that* (12) *holds true for some constants*

η, $C>0$ independent of n (by Lemma 1), so that there exist a sequence $\{(s_n, y_n^1)\}\subset \boldsymbol{R}\times \boldsymbol{R}^N$ such that

$$(13)\qquad u_n(\cdot+s_n, \cdot+y_n^1)\to u^1\neq 0 \qquad as\ n\to\infty$$

weakly in $L^\infty(\boldsymbol{R}; H^1)$ and strongly in $C(I; L^2(\Omega))$, where $I\Subset \boldsymbol{R}$ and $\Omega\Subset \boldsymbol{R}^N$.*

Lemma 3. *$\{u_n\}$ is a uniformly bounded sequence in $L^\sigma(I\times \boldsymbol{R})$ for any interval I in $\boldsymbol{R}$ and we have that $u_n\to u^1$ a.e. $I\times \boldsymbol{R}^N$ (by Lemma 2). Then,*

$$(14)\qquad |u_n|^{4/N}u_n-|u_n-u^1|^{4/N}(u_n-u^1)-|u^1|^{4/N}u^1\to 0 \qquad as\ n\to\infty$$

in $L^{\sigma'}(I\times \boldsymbol{R}^N)$, and

$$(15)\qquad \lim_{n\to\infty}\int_I\left(\int_{\boldsymbol{R}^N}||u_n|^\sigma-|u_n-u^1|^\sigma-|u^1|^\sigma|dx\right)dt=0,$$

where $1/\sigma+1/\sigma'=1$.

Lemma 4. *Put $u_n^1\equiv u_n(\cdot+s_n, \cdot+y_n^1)$. $u^1\in C(\boldsymbol{R}; H^1)$ is a solution of (LP) and satisfies*

$$(16)\qquad \lim_{n\to\infty}\int_I\{E_{\sigma,\lambda}(u_n^1)-E_{\sigma,\lambda}(u_n^1-u^1)-E_{\sigma,\lambda}(u^1)\}dt=0$$

for $I\Subset \boldsymbol{R}$.

Suppose $\lim_{n\to\infty}|u_n^1-u^1|_{\sigma,\infty}\neq 0$. At this stage, we consider $f_n^2\equiv u_n^1-u^1$ which also forms a bounded sequence in $L^\infty(\boldsymbol{R}; H^1)$. It is worth while to note that f_n^2 enjoys the property

$$(17)\qquad 2i\partial_t f_n^2+\Delta f_n^2+\lambda|f_n^2|^{4/N}f_n^2\to 0$$

weakly* in $L^\infty(\boldsymbol{R}; H^{-1})$ as $n\to\infty$. Repeating the above argument, we obtain the main assertion of Theorem A. We also have

Proposition B. *If u is a global solution of (LP) such that $u\in C(\boldsymbol{R}; H^1)$, then $E_{\sigma,\lambda}(u)\geqq 0$.*

Thus we can complete the proof of Theorem A.

Remarks. 1. Theorem A is closely related to a phenomenon which has been observed in various nonlinear problems by the name of bubble theorem or concentration-compactness theorem (for example, see [1], [4] and their references).

2. (B.1) suggests that blow-up solutions may exist beyond the blow-up time in some sense.

3. Lemma 2 is a space-time version of Lieb [3; Lemma 6], and Lemma 3 is a variant of Brézis-Lieb [2].

4. The proof of Theorem A is inspired by the work of Brézis-Coron [1]. One may find the idea of it in [4].

References

[1] Brézis, H. and Coron, J. M.: Convergence of H-system or how to blow bubbles. Arch. Rat. Mech. Anal., **82**, 313–376 (1983).

[2] Brézis, H. and Lieb, E. H.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc., **88**, 486–490 (1983).

[3] Lieb, E. H.: On the lowest eigenvalue of Laplacian for the intersection of two domains. Invent. Math., **74**, 441–448 (1983).

[4] Nawa, H.: Mass concentration phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity. II (to appear in Kodai Math. J., **13** (1990)).