47. Symmetries of the Garnier System and of the Associated Polynomial Hamiltonian System

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Introduction. The aim of this note is to present a group of symmetries for the Garnier system, which is a system of partial differential equations obtained by monodromy preserving deformations of second order Fuchsian differential equations on $\boldsymbol{P}^1$, and for the associated polynomial Hamiltonian system.

Let us consider a Pfaffian system:

$$E(\theta): \qquad dx_i=\sum_{j=1}^{n} G_{ij}(x,t,\theta)dt_j \qquad i=1,\cdots,m,$$

where $G_{ij}(x,t,\theta)$ are rational functions in $(x,t)=(x_1,\cdots,x_m,t_1,\cdots,t_n)$ depending on parameters $\theta\in\boldsymbol{C}^N$ and d stands for the exterior differentiation with respect to (x,t). For a birational transformation $S:(x,t)\to(x',t')$, we denote by $S\cdot E(\theta)$ the system of differential equations in the variables (x',t') obtained from $E(\theta)$ by the transformation S.

Definition. A *group of symmetries* for the system E is a group whose element is a pair $\sigma=(S,l)$ of a birational transformation $S:(x,t)\to(x',t')$ and an affine transformation $l:\boldsymbol{C}^N\to\boldsymbol{C}^N$ such that $S\cdot E(\theta)=E(l(\theta))$.

Let $\sigma=(S,l)$ and $\sigma'=(S',l')$ be symmetries of the system E. The product $\sigma\cdot\sigma'$ and the inverse σ^{-1} to σ are defined by $\sigma\cdot\sigma':=(S\circ S',l\circ l')$ and $\sigma^{-1}:=(S^{-1},l^{-1})$, respectively.

1. Garnier system $\mathcal{G}_n$ and the associated system $\mathcal{H}_n$. The n-dimensional Garnier system is the Hamiltonian system

$$\mathcal{G}_n: \qquad d\lambda_i=\sum_{j=1}^{n}\{K_j,\lambda_i\}dt_j, \qquad d\mu_i=\sum_{j=1}^{n}\{K_j,\mu_i\}dt_j,$$

$i=1,\cdots,n$, where $\{\cdot,\cdot\}$ stands for the Poisson bracket

$$\{f,g\}=\sum_i\left(\frac{\partial f}{\partial\mu_i}\frac{\partial g}{\partial\lambda_i}-\frac{\partial g}{\partial\mu_i}\frac{\partial f}{\partial\lambda_i}\right).$$

The Hamiltonians $K_i=K_i(\theta,\lambda,\mu,t)$ are given by

$$K_i=M_i\sum_{k=1}^{n}M^{k,i}\left\{\mu_k^2-\sum_{m=1}^{n+2}\frac{\theta_m-\delta_{im}}{\lambda_k-t_m}\mu_k+\frac{\kappa}{\lambda_k(\lambda_k-1)}\right\},$$

where $\theta_1,\cdots,\theta_{n+2}$, $\kappa:=(1/4)(\sum_{i=1}^{n+2}\theta_i-1)^2-(1/4)\theta_\infty^2$ are constants, $t_{n+1}=0$, $t_{n+2}=1$, and

$$M_i=-\frac{\Lambda(t_i)}{T'(t_i)}, \qquad M^{k,i}=\frac{T(\lambda_k)}{(\lambda_k-t_i)\Lambda'(\lambda_k)},$$

$(i,k=1,\cdots,n,n+1,n+2)$ defined by using

$$T(x)=\prod_{i=1}^{n+2}(x-t_i), \qquad \Lambda(x)=\prod_{k=1}^{n}(x-\lambda_k).$$

Note that $\mathcal{G}_1$ is the sixth Painlevé system and hence the Garnier system can be thought of a generalization of the sixth Painlevé system.

We transform $\mathcal{G}_n$ into the Hamiltonian system with the Painlevé property, i.e., the system of which solutions are free of branching singularity whose position depends on initial conditions.

Proposition 1 [3]. *By the change of variables* ϕ: $(\lambda, \mu, t)\to(q, p, s)$ *defined by*

$$q_i=-t_iM_i, \quad p_i=-(t_i-1)\sum_k \frac{M^{k,i}\mu_k}{\lambda_k(\lambda_k-1)}, \quad s_i=\frac{t_i}{t_i-1},$$

$i=1, \cdots, n,$ *the Garnier system* $\mathcal{G}_n$ *is transformed into the system enjoying the Painlevé property*:

$$\mathcal{H}_n: \qquad dq_i=\sum_j \{H_j, q_i\}ds_j, \qquad dp_i=\sum_j \{H_j, p_i\}ds_j,$$

$i=1, \cdots, n,$ *with Hamiltonians*:

$$H_i :=\frac{1}{s_i(s_i-1)}\left[\sum_{j,k=1}^{n} E_{ijk}(s, q)p_jp_k-\sum_{j=1}^{n} F_{ij}(s, q)p_j+\kappa q_i\right],$$

where E_{ijk}, $F_{ij}\in \boldsymbol{C}(s)[q]$.

As for the explicit form of E_{ijk} and $F_{ij}\in \boldsymbol{C}(s)[q]$, see [3].

2. Symmetries of the Garnier system. Let $V=\{\theta=(\theta_1, \cdots, \theta_{n+2}, \theta_\infty)\in \boldsymbol{C}^{n+3}\}$ be the space of parameters of the systems $\mathcal{G}_n$ and $\mathcal{H}_n$; and let $\mathcal{G}_n(\theta)$ and $\mathcal{H}_n(\theta)$ be the systems $\mathcal{G}_n$ and $\mathcal{H}_n$ with fixed parameters θ, respectively.

We give a group of symmetries for the Garnier system $\mathcal{G}_n$. Let S_m: $(\lambda, \mu, t)\to(\lambda', \mu', t')$ $(m=1, \cdots, n+2)$ be birational transformations defined as follows.

$$S_m(m=1, \cdots, n): \qquad \lambda_i'=\frac{t_m-\lambda_i}{t_m-1}, \qquad \mu_i'=-(t_m-1)\mu_i,$$

$$t_i'=\begin{cases}\dfrac{t_m-t_i}{t_m-1}, & (i\neq m, n+1),\\[2mm] \dfrac{t_m}{t_m-1}, & (i=m);\end{cases}$$

$$S_{n+1}: \qquad \lambda_i'=1-\lambda_i, \quad \mu_i'=-\mu_i, \quad t_i'=1-t_i;$$

$$S_{n+2}: \qquad \lambda_i'=\frac{\lambda_i}{\lambda_i-1}, \quad \mu_i'=-(\lambda_i-1)^2\mu_i-\alpha(\lambda_i-1), \quad t_i'=\frac{t_i}{t_i-1}.$$

where $\alpha=\frac{1}{2}(\theta_1+\cdots+\theta_{n+2}-1+\theta_\infty)$. Let l_m: $V\to V$ be linear transformations defined by

$$l_m(m=1, \cdots, n): \quad (\theta_1, \cdots, \theta_m, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2}, \theta_\infty)$$
$$\longmapsto(\theta_1, \cdots, \theta_{m-1}, \theta_{n+1}, \theta_{m+1}, \cdots, \theta_n, \theta_m, \theta_{n+2}, \theta_\infty),$$
$$l_{n+1}: \quad (\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2}, \theta_\infty)\longmapsto(\theta_1, \cdots, \theta_n, \theta_{n+2}, \theta_{n+1}, \theta_\infty),$$
$$l_{n+2}: \quad (\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2}, \theta_\infty)\longmapsto(\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_\infty, \theta_{n+2}).$$

Set $\sigma_m :=(S_m, l_m)$ $(m=1, \cdots, n+2)$ and let $L :=\langle\sigma_1, \cdots, \sigma_{n+2}\rangle$ be the group generated by σ_m's. The group L is isomorphic to the symmetric group on $n+3$ letters.

Furthermore, let $T_m: (\lambda, \mu, t) \to (\lambda', \mu', t')$ $(m=1, \cdots, n+2)$ be the birational transformations defined by

$$\lambda_i' = \lambda_i, \quad \mu_i' = \mu_i - \frac{\theta_m}{\lambda_i - t_m}, \quad t_i' = t_i \quad (i=1, \cdots, n),$$

and let $h_m: V \to V$ $(m=1, \cdots, n+2, \infty)$ be the affine transformations given by

$$h_m: \quad (\theta_1, \cdots, \theta_m, \cdots, \theta_\infty) \longmapsto (\theta_1, \cdots, -\theta_m, \cdots, \theta_\infty).$$

Set $\tau_m = (T_m, h_m)$ and $\tau_\infty = (id., h_\infty)$; they generate a group isomorphic to $(Z/2Z)^{n+3}$. Let $G := \langle \sigma_1, \cdots, \sigma_{n+2}, \tau_1, \cdots, \tau_{n+2}, \tau_\infty \rangle$ be the group generated by σ_i's and τ_m's.

Theorem 2. *The group G is a group of symmetries for the Garnier system $\mathcal{G}_n$ and is isomorphic to the Weyl group of type B_{n+3}.*

3. Symmetries of $\mathcal{H}_n$. We shall present a group of symmetries for the Hamiltonian system $\mathcal{H}_n$ which is naturally induced from the group G. For each element $\sigma = (S, l)$ of the group G, we seek for a birational transformation $\hat{S}: (q, p, s) \to (q', p', s')$ such that the following diagram commutes:

$$\begin{tikzcd}
(\lambda, \mu, t) \arrow[r, "S"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & (\lambda', \mu', t') \arrow[d, "\phi"'] \\
(q, p, s) \arrow[r, "\hat{S}"] & (q', p', s')
\end{tikzcd}$$

where the right and left vertical arrows indicate the transformations ϕ, given in Proposition 1, which take the Garnier system $\mathcal{G}_n(\theta)$ into the Hamiltonian system $\mathcal{H}_n(\theta)$.

Theorem 3. (i) *For each element $\sigma = (S, l) \in G$, there exists a unique birational transformation $\hat{S}: (q, p, s) \to (q', p', s')$ which make the above diagram commute.*

(ii) *The group $\hat{G} = \langle \hat{\sigma}_1, \cdots, \hat{\sigma}_{n+2}, \hat{\tau}_1, \cdots, \hat{\tau}_{n+2}, \hat{\tau}_\infty \rangle$, generated by $\hat{\sigma}_m = (\hat{S}_m, l_m)$, $\hat{\tau}_m := (\hat{T}_m, h_m)$ $(i=1, \cdots, n+2)$ and $\tau_\infty := (id., h_\infty)$ is a group of symmetries for the system $\mathcal{H}_n$ and is isomorphic to the Weyl group of type B_{n+3}.*

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