96. Orbi-maps and 3-orbifolds

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1. Definitions. An n-orbifold is a topological space locally homeomorphic to (an open set in $\boldsymbol{R}^n$)/(a finite group action) and each point of it is provided with an isotropy data. By the symbol $|X|$, we shall mean the underlying space of the orbifold X.

For studying orbifolds, we need a map between orbifolds which respects their orbifold structures. An orbifold X is *good* if $|X|$ is homeomorphic to (a manifold $\tilde{X}$)/(a properly discontinuous action). In this paper, orbifolds which we deal with are good orbifolds. All orbifolds will be assumed to be good unless otherwise specified. If $\tilde{X}$ is simply connected, the quotient map $p: |\tilde{X}| \to |X|$ is called the *universal orbi-covering*.

Let X and Y be orbifolds. Let $p: |\tilde{X}| \to |X|$ and $q: |\tilde{Y}| \to |Y|$ be the universal orbi-coverings. We introduce an orbi-map between X and Y as follows; By an *orbi-map* $f: X \to Y$, we shall mean a continuous map $h: |X| \to |Y|$ with a fixed continuous map $\tilde{h}: \tilde{X} \to \tilde{Y}$ which satisfies the following conditions:

(01) $$h \circ p = q \circ \tilde{h}.$$

(02) For each $\sigma \in \mathrm{Aut}(\tilde{X}, p)$, there exists a $\tau \in \mathrm{Aut}(\tilde{Y}, q)$ such that $\tilde{h} \circ \sigma = \tau \circ \tilde{h}$.

(03) There exists a point $\tilde{x} \in \tilde{X} - p^{-1}(\Sigma X)$ such that $\tilde{h}(\tilde{x}) \in \tilde{Y} - q^{-1}(\Sigma Y)$.

2. Constructions and modifications of orbi-maps.

2.1. Theorem. *Let M be a compact 2- or 3-orbifold and N an orientable 3-orbifold such that the total space of the universal orbi-covering of* $\mathrm{Int}(N)$ *is homeomorphic to $\boldsymbol{R}^3$. Suppose $\varphi: \pi_1(M) \to \pi_1(N)$ is a homomorphism such that for any local group G_x of M, $\varphi(G_x) \not\cong A_5$. Then, there exists an orbi-map $f: M \to N$ such that $f_* = \varphi$.*

2.2. Theorem (*Transversal modification of dimension* 3). *Suppose M and N are compact 3-orbifolds such that N is containing a properly embedded, 2-sided, 2-suborbifold F such that* $\mathrm{Ker}(\pi_1(F) \to \pi_1(N)) = 1$, $\pi_2(F) = 0$, *and the total space of the universal orbi-covering of* $\mathrm{Int}(N-F)$ *is homeomorphic to $\boldsymbol{R}^3$. Suppose $f: M \to N$ is any orbi-map such that for any local group G, $f_*(G) \not\cong A_5$. Then there exists an orbi-map $g: M \to N$ such that*

(1) *g is C-equivalent to f,*

(2) *each component of $g^{-1}(F)$ is a properly embedded, 2-sided, incompressible 2-suborbifold in M, and*

(3) *for properly choosen product neighborhoods $F \times [-1, 1]$ of $F = F \times 0$ in N and $g^{-1}(F) \times [-1, 1]$ of $g^{-1}(F) = g^{-1}(F) \times 0$ in M, g maps each fiber $x \times$*

$[-1, 1]$ *homeomorphically to the fiber* $g(x)\times[-1, 1]$ *for each* $x \in g^{-1}(F)$.

As an application of these theorems, we can prove the "Untangling theorem." (See 4.2.)

3. The classification theorem. Let M and N be 3-orbifolds. Let $\Psi: \pi_1(M, x)\to\pi_1(N, y)$ be a homomorphism. We say that Ψ *respects the peripheral structure,* if the following holds. For each boundary component F of M, there exists a boundary component G of N, such that $\Psi(i_*(\pi_1(F, x')))\subset A$, and A is conjugate in $\pi_1(N, y)$ to $j_*(\pi_1(G, y'))$, where i and j are inclusions.

Let $\mathcal{W}$ be the class of all compact, connected, orientable 3-orbifolds which are

(i) finitely uniformizable,

(ii) irreducible,

(iii) sufficiently large,

(iv) the local group of any point is not isomorphic to A_5, and

(v) in which every turnover with non-positive Euler number is boundary-parallel.

By the recent result of W.D. Dunbar [1], the orbifold M which belongs to $\mathcal{W}$ has a hierarchy. With these preparation, we can prove the classification theorem as parallel as the proof of 13.6 of [2].

3.1. Theorem. *Let* M, $N\in\mathcal{W}$. *Suppose all the components of* ∂M *are incompressible in* M. *Let* $\Psi: \pi_1(M)\to\pi_1(N)$ *be an isomorphism which respects the peripheral structure. Then either*

(1) *there exists an orbi-isomorphism* $f: M\to N$ *which induces* Ψ

or

(2) *M is a twisted I-bundle over a closed non-orientable 2-orbifold F and N is an I-bundle over a 2-orbifold G such that* $\pi_1(F)\cong\pi_1(G)$.

As the corollary of it, we derive

3.2. Corollary. *Let* M, $N\in\mathcal{W}$. *Suppose M is closed and* $\pi_1(M)\cong\pi_1(N)$. *Then M and N are orbi-isomorphic.*

Zimmermann showed the consequence including 3.2 in [6] by using Thurston's orbifold geometrization theorem and his own theorem.

4. Applications for links and tangles. We apply our result so that we classify a class of links. Recall that a link (S^3, L) is *prime* if there is no S^2 in S^3 that separates the component of L, and any S^2 that meets L in two points, transversely, bounds in S^3 one and only one ball intersecting L in a single unknotted spanning arc. Let (S^3, L) be a link and X be the orbifold with $\Sigma X=L$. Suppose the orders of every local group of ΣX are n. We call such an orbifold X *the associated orbifold with weight n of* (S^3, L), denoted by $O_{(L,n)}$. We define the *n-weighted orbi-invariant* of (S^3, L), denoted by $\mathrm{Orb}_n(L)$, by the fundamental group of the associated orbifold with weight n of (S^3, L). We call the link (S^3, L) *sufficiently large* if the orbifold $O_{(L,*)}$ is sufficiently large. By applying 3.2, we have the following result.

4.1. Theorem. *Suppose* (S^3, L) *and* (S^3, L') *are prime and sufficiently large links.* (S^3, L) *and* (S^3, L') *are the same link type, if and only if*

$\mathrm{Orb}_n(L) \cong \mathrm{Orb}_n(L')$ *for some* $n \in \boldsymbol{Z}^{\pm}$, $n \geqq 2$.

Zimmermann shows the similar result in [7].

For a tangle (B, t), we define $\mathrm{Orb}_n(t)$ in the similar manner as in the case of links. We get the following result as an application of 2.1 and 2.2.

4.2. Theorem. *Let* (B, t) *be a k-strings tangle.* (B, t) *is the trivial tangle if and only if* $\mathrm{Orb}_2(t) \cong A_1 * \cdots * A_k$, *where* $A_i \cong \boldsymbol{Z}_2$ *for each i.*

References

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