86. *Weyl's Type Criterion for General Distribution Mod 1*

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1. In 1916, Weyl [6] has proposed the following necessary and sufficient condition for uniform distribution mod 1 of the real number sequences which is now called *Weyl's criterion*: *The sequence* (x_n), $(n=1, 2, 3, \cdots)$ *is uniformly distributed* mod 1, *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i\nu x_n}=0$$

for any natural number ν.

Schoenberg [5], in 1928, first generalized the concept of uniform distribution mod 1 to that of general distribution mod 1 (or asymptotic distribution mod 1) and obtained many interesting results including generalizations of Weyl's criterion. Later many mathematicians have also proposed various forms of criteria for general distribution mod 1 (cf. [3]). But their criteria were not of exponential sum type. Among them, Helmberg [2] has made an interesting contribution to this field with view points concerning mainly numerical computations of the integrals of type: $\int_0^\infty f(x)dx$. Recently the first author found a natural generalization of Weyl's criterion (Theorem 1) which seems new to the authors. In this note, we give this criterion for generally distributed mod 1 sequences with some applications: estimations of some trigonometric sums, a generalization of Erdös-Turán's theorem and a generalization of LeVeque's inequality. The proof of these results and other results in various directions will be given elsewhere.

2. **Definition.** Let $\mu(x)$ be a distribution function with $\mu(0)=0$, $\mu(1)=1$, $d\mu(x)=w(x)dx$ where $w(x)$ is the density function of $\mu(x)$ satisfying the following conditions: (a) $w(x)$ is piecewise continuous on $[0, 1]$ and $0<w(x)<+\infty$. (b) The number of the discontinuity points of $w(x)$ is finite. (c) ${}^\#\{z\in[0,1]\,|\,\lim_{x\to z-0}w(x)=0 \text{ or } \lim_{x\to z+0}w(x)=0\}<\infty$, where ${}^\#A$ denotes the number of elements of the set A. Then, the real number sequence $(x_n)_{n=1}^{\infty}$ is called *μ-distributed* mod 1, if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi([0,x):\{x_n\})=\mu(x)$$

for each $x\in[0,1]$, where $\chi([0,x):u)$ is the indicator function of $[0,x)$ and $\{x\}$ denotes the fractional part of x.

3. **Results.** Then we have the main theorem.

Theorem 1. *The sequence* (x_n) *is μ-distributed* mod 1, *if and only if*

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$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{e^{2\pi i\nu x_n}}{w(\{x_n\})}=\delta_{\nu,0}$$

for all $\nu\in\boldsymbol{Z}$, *where* $\delta_{\nu,\mu}$ *is Kronecker's delta.*

From Theorem 1, we have the following two examples of estimations of trigonometric sums.

(1) $$\sum_{n\leqslant x}\sqrt{1-\{\sin(n\theta)\}^2}\,e^{2\pi i\nu\sin(n\theta)}=o(x)$$

for any $\nu\in\boldsymbol{N}$ and any $\theta\in\boldsymbol{Q}$.

(2) $$\sum_{n\leqslant x}2^{\{n\alpha\}}e^{2\pi i\nu 2^{\{n\alpha\}}}=o(x)$$

for any $\nu\in\boldsymbol{N}$ and any irrational α, where $o(\cdot)$ denotes Landau's small o-symbol.

Next, we give the following generalizations of Erdös-Turán's theorem and LeVeque's inequality.

Theorem 2. *For any* $m\in\boldsymbol{N}$, *any sequence* $\xi=\{x_1,x_2,\cdots,x_N\}$ *with* $0\leqq x_j<1$ *and any distribution function* $\mu(x)$, *we have*

$$D_N\leqslant C_1\Big(\frac{6}{m+1}+\frac{4}{\pi}\sum_{\nu=1}^{m}\Big(\frac{1}{\nu}-\frac{1}{m+1}\Big)\Big|\frac{1}{N}\sum_{n=1}^{N}\frac{e^{2\pi i\nu x_n}}{w(x_n)}-\Big(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}-1\Big)\Big|$$
$$+6\Big|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}-1\Big|+4\Big|\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)}-\frac{1}{2}\Big|\Big),$$

where

$$D_N=D_N(\xi\,;\mu)=\sup_{0\leqslant a<b\leqslant 1}\Big|\frac{1}{N}\sum_{n=1}^{N}\chi([a,b):x_n)-(\mu(b)-\mu(a))\Big|$$

is the discrepancy of the sequence ξ *with respect to* $\mu(x)$ *and* C_1 *is a computable constant depending only on* $\mu(x)$.

Theorem 3. *For any* $m\in\boldsymbol{N}$, *any sequence* $\xi=\{x_1,x_2,\cdots,x_N\}$ *with* $0\leqq x_j<1$ *and any distribution function* $\mu(x)$, *we have*

$$D_N\leqslant C_2\Big(\frac{6}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\Big|\frac{1}{N}\sum_{n=1}^{N}\frac{e^{2\pi i\nu x_n}}{w(x_n)}-\Big(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}-1\Big)\Big|^2$$
$$+4\max\Big(\Big(\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)}-\frac{1}{2}\Big)^3,0\Big)$$
$$-4\min\Big(\Big(\Big(\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)}-\frac{1}{2}\Big)-\Big(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{w(x_n)}-1\Big)\Big)^3,0\Big)\Big)^{1/3},$$

where C_2 *is a computable constant depending only on* $\mu(x)$.

In these theorems, the extra term:

$$\frac{1}{N}\sum_{n=1}^{N}\frac{x_n}{w(x_n)}-\frac{1}{2}$$

appears. In case of uniform distribution mod 1, this term does not appear due to the invariance of the discrepancy extended over all half-open intervals mod 1 under the translation: $T(x_n)=x_n+c$ ([3], p. 114).

References

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