82. *Mordell-Weil Lattices and Galois Representation.* III

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5. Galois representation arising from the Mordell-Weil lattices. To explain the basic idea, let us consider the "elementary" situation. Let E be an elliptic curve defined over $\boldsymbol{Q}(t)$, t being a variable over $\boldsymbol{Q}$, and let $f: S \to \boldsymbol{P}^1$ be its Kodaira-Néron model, which is an elliptic surface defined over $\boldsymbol{Q}$. Assume as before that f is not smooth. Letting $\bar{\boldsymbol{Q}}$ be the algebraic closure of $\boldsymbol{Q}$ and $K=\bar{\boldsymbol{Q}}(t)$, consider the Mordell-Weil group of K-rational points $E(K)$. Obviously the Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ acts on $E(K)$, and it makes the height pairing stable. Thus we have the Galois representation on the Mordell-Weil lattice $E(K)/(\mathrm{tor})$ or $E(K)^0$: let

(5.1) $$\rho: \ \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{Aut}(E(K)^0).$$

There arises a natural question:

Question 5.1. (1) *How big can* $\mathrm{Im}(\rho)$ *be? and:*

(2) *How small can* $\mathrm{Im}(\rho)$ *be?*

The interest of the first question is obvious. The second one is also interesting, because if the image of ρ is trivial, then we have $E(\bar{\boldsymbol{Q}}(t))=E(\boldsymbol{Q}(t))$ so that the rank of E over $\boldsymbol{Q}(t)$ can be relatively big. The intermediate case can be also of some interest (e.g. [5]).

Suppose, for instance, that $S\otimes\bar{\boldsymbol{Q}}$ is a rational elliptic surface without reducible fibres. Then, by Theorem 2.1, the Mordell-Weil lattice is the root lattice E_8, and hence we have

(5.2) $$\rho: \ \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{Aut}(E_8)=W(E_8).$$

The Hasse zeta function of the surface S over $\boldsymbol{Q}$ is given by

(5.3) $$\zeta(S/\boldsymbol{Q}, s)=\zeta(s)\zeta(s-1)^2\zeta(s-2)L(\rho, s-1)$$

(up to finitely many Euler factors) where $L(\rho, s)$ is the Artin L-function attached to ρ and $\zeta(s)$ is the Riemann zeta function.

Now the first question asks: is it possible to have $\mathrm{Im}(\rho)=W(E_8)$ for some $E/\boldsymbol{Q}(t)$? We can affirmatively answer this question (Theorem 7.1) and its variant for E_7, E_6, etc. Thus we obtain infinitely many Galois extensions of $\boldsymbol{Q}$ with Galois group $W(E_8)$, having a natural representation ρ on the lattice E_8. Since $W(E_8)$ contains a subgroup H of index 2 such that $H/\{\pm 1\}$ is a simple group ([1, Ch. 6]), $L(\rho, s)$ is essentially of non-abelian type.

Our results also answer the question, first remarked by Weil [9, p. 558] and then studied by Manin [4, Ch. 4] in more detail, concerning the image of the Galois representation arising from the 27 lines on a smooth cubic surface or more generally from the exceptional curves on a Del Pezzo surface (cf. Remark 6.3).

Now, for the second question as to the case Im (ρ) trivial, we have also a satisfactory answer. We obtain 8 parameter family of elliptic curves E_λ over $\boldsymbol{Q}(t)$, having 8 free generators of $E_\lambda(\boldsymbol{Q}(t))$ which depend rationally on the parameter λ (Theorem 7.2).

Needless to say, the generalization of these results to the case of irrational elliptic surfaces (e.g. $K3$ surfaces) with high rank will be extremely interesting.

6. Monodromy of the Mordell-Weil lattices. To fix the idea, we consider the family of elliptic curves $\{E_\lambda\}$ defined by the Weierstrass equation:

$$y^2=x^3+\left(\sum_{i=0}^{3} p_i t^i\right)x+\left(\sum_{i=0}^{3} q_i t^i+t^5\right), \tag{6.1}$$

where $\lambda=(p_0, \cdots, p_3, q_0, \cdots, q_3) \in A^8$, the 8-dimensional affine space over $\boldsymbol{Q}$, and t is a variable over the function field $\boldsymbol{Q}(A^8)$. For any λ, E_λ is an elliptic curve defined over $\boldsymbol{Q}(\lambda)(t)$. Let S_λ denote the Kodaira-Néron model of E_λ over $\boldsymbol{Q}(\lambda)(t)$. It is a smooth algebraic surface defined over $\boldsymbol{Q}(\lambda)$ such that $S_\lambda \otimes \overline{\boldsymbol{Q}(\lambda)}$ is a rational surface, where $\overline{\boldsymbol{Q}(\lambda)}$ denotes the algebraic closure of $\boldsymbol{Q}(\lambda)$. Since S_λ has a singular fibre of type II over $t=\infty$, we have the specialization map $sp_\infty: E(\overline{\boldsymbol{Q}(\lambda)}(t)) \to \overline{\boldsymbol{Q}(\lambda)}$ (Lemma 3.3).

Theorem 6.1. *For generic λ (i.e. for $p_0, \cdots, q_3$ independent variables) over $\boldsymbol{Q}$, let $P_1, \cdots, P_8$ be a basis of $E_\lambda(\overline{\boldsymbol{Q}(\lambda)}(t)) \simeq E_8$, and let $u_i=sp_\infty(P_i)$. Then the map*

$$\rho_\lambda: \ \mathrm{Gal}\,(\overline{\boldsymbol{Q}(\lambda)}/\boldsymbol{Q}(\lambda)) \longrightarrow W(E_8) \tag{6.2}$$

is surjective. If $\mathfrak{K}_\lambda$ is the Galois extension of $\boldsymbol{Q}(\lambda)$ which correspond to Ker (ρ_λ), *then $\mathfrak{K}_\lambda=\boldsymbol{Q}(\lambda)(u_1, \cdots, u_8)=\boldsymbol{Q}(u_1, \cdots, u_8)$. Therefore $\boldsymbol{Q}(u_1, \cdots, u_8)/\boldsymbol{Q}(p_0, \cdots, q_3)$ is a Galois extension with Galois group $W(E_8)$.*

Sketch of the proof. To show the surjectivity of ρ_λ, it suffices to prove it after we make the base extension of $\boldsymbol{Q}$ to $\boldsymbol{C}$. Then we observe that, for any (specialized) $\lambda \in \boldsymbol{C}^8$, the following are equivalent: (i) the Mordell-Weil lattice $E_\lambda(\boldsymbol{C}(t)) \simeq E_8$, (ii) S_λ has no reducible fibre and (iii) the affine surface S'_λ defined by (6.1) is smooth. Hence the locus of $\lambda \in \boldsymbol{C}^8$ having Mordell-Weil lattice E_8 is $\boldsymbol{C}^8-D$, D being the discriminant locus of (6.1). Now the family (6.1) can be viewed as the semi-universal deformation of the rational double point of type E_8: $y^2=x^3+t^5$, parametrized by A^8. Then, by the well-known result in the singularity theory due to Brieskorn, Tjurina and others (see e.g. [2], [7]), the monodromy representation

$$\pi_1(\boldsymbol{C}^8-D) \longrightarrow W(E_8)$$

is surjective, which proves the assertion.

Remark 6.2. As the above sketched proof shows, the monodromy and "degeneration" of the Mordell-Weil lattices are closely related to the deformation of certain isolated singularities, not necessarily of rational double points.

Remark 6.3. Also there is a close connection between certain class of rational elliptic surfaces and the Del Pezzo surfaces. It seems that the geometry of the latter (cf. [4]), e.g. the exceptional curves on such, can be much better understood in terms of the Mordell-Weil lattices. For instance, a rational elliptic surface S with Mordell-Weil lattice of type E_6 is the blowing-up of three infinitely near points of smooth cubic surfaces V_1 and V_2, and the 54 minimal sections are exactly the pull-back of the 27 lines on each V_i. We can explicitly write down "the equation of 27 lines" this way. Similarly, we have an algebraic equation of degree 240 whose roots are the 240 "roots" of the root lattice E_8!

7. Arithmetic consequences. The first consequence. By Hilbert's irreducibility theorem ([3, Ch. 9]), we have:

Theorem 7.1. *There exist infinitely many* $\lambda=(p_0, \cdots q_3) \in \boldsymbol{Q}^8$ *with the following properties (let* $\mathfrak{K}_\lambda$ *be defined as before):*

1) $\mathfrak{K}_\lambda/\boldsymbol{Q}$ *is a Galois extension with Galois group* $W(E_8)$.

2) *The elliptic curve* E_λ *over* $\boldsymbol{Q}(t)$ *has the Mordell-Weil group* $E_\lambda(\bar{\boldsymbol{Q}}(t)) = E_\lambda(\mathfrak{K}_\lambda(t)) \simeq E_8$ *and* $E_\lambda(\boldsymbol{Q}(t))=0$.

3) *The Hasse zeta function of the surface* S_λ *over* $\boldsymbol{Q}$ *is given by* (5.3) *in which the Artin L-function* $L(\rho_\lambda, s)$ *is essentially of non-abelian type.*

It seems possible to give explicit numerical examples as above. (N.B. The mere existence of infinitely many Galois extensions with Galois group $W(E_8)$ (or any finite reflection group) is trivial in view of Chevalley's theorem and Hilbert's irreducibility theorem).

The second consequence. For a moment, let $\lambda=(p_0, \cdots, p_3, q_0, \cdots, q_3)$ be generic again. Then, with the notation of Theorem 6.1, $u_1, \cdots, u_8$ are independent variables over $\boldsymbol{Q}$. The ring of $W(E_8)$-invariants in $\boldsymbol{Q}[u_1, \cdots, u_8]$ is a graded polynomial ring with generators of degree 2, 8, 12, 14, 18, 20, 24 and 30 ([1, Ch. 6]). Indeed, we have

$$\boldsymbol{Q}[u_1, \cdots, u_8]^{W(E_8)} = \boldsymbol{Q}[p_0, \cdots, p_3, q_0, \cdots, q_3] \tag{7.1}$$

and we can uniquely write

$$p_i = I_{20-6i}(u_1, \cdots, u_8), \quad q_i = I_{30-6i}(u_1, \cdots, u_8) \quad (0 \leq i \leq 3) \tag{7.2}$$

where $I_w(u)$ are polynomials of degree w in $u_1, \cdots, u_8$ with $\boldsymbol{Q}$-coefficients. On the other hand, let $u_j = sp_\infty(P_j)$ $(j=1, \cdots, 240)$ be all the "roots": each u_j is a $\boldsymbol{Z}$-linear combination of $u_1, \cdots, u_8$. Let $\delta(u) = \Pi_{i<j}(u_i - u_j)$. With these notation, we have

Theorem 7.2. *For any* $u=(u_1, \cdots, u_8) \in \boldsymbol{Q}^8$ *such that* $\delta(u) \neq 0$, *define* $p_i, q_i \in \boldsymbol{Q}$ *by* (7.2) *and let* $\lambda=(p_0, \cdots, p_3, q_0, \cdots, q_3) \in \boldsymbol{Q}^8$. *Then the elliptic curve* E_λ *over* $\boldsymbol{Q}(t)$, *defined by* (6.1), *has the Mordell-Weil group* $E_\lambda(\boldsymbol{Q}(t))$ *of rank* 8 *(without any constant field extension). Moreover there is a basis* $\{P_1, \cdots, P_8\}$ *of* $E_\lambda(\boldsymbol{Q}(t))$, *of the form* $P_i=(x_i, y_i)$ *with*

$$x_i = u_i^{-2}t^2 + a_i t + b_i, \qquad y_i = u_i^{-3}t^3 + c_i t^2 + d_i t + e_i, \tag{7.3}$$

which depend rationally on the parameters $u_1, \cdots, u_8$.

Corollary 7.3. *Fix* $u=(u_1, \cdots, u_8) \in \boldsymbol{Q}^8$ *and* $\lambda=(p_0, \cdots, q_3) \in \boldsymbol{Q}^8$ *as above. Then, specializing t to any rational number (with only finitely many exception), we obtain a family of elliptic curves* $E^{(t)}$, *given with* $\boldsymbol{Q}$*-rational points* $P_1^{(t)}, \cdots, P_8^{(t)}$, *such that*

$$(7.4)\qquad \lim_{h(t)\to\infty} \det(\langle P_i^{(t)}, P_j^{(t)}\rangle_{\mathrm{can}}/h(t)) = 2^{-8}$$

where $h(t)$ is the standard height of a point on $\boldsymbol{P}^1_{\boldsymbol{Q}}$ and $\langle\ ,\ \rangle_{\mathrm{can}}$ is the canonical height on the Mordell-Weil group of an elliptic curve over $\boldsymbol{Q}$ (cf. [8]).

This follows from Tate's theorem (loc. cit) and $\det(E_8)=1$. The factor 2^8 is caused by our definition of the height pairing (the part I) which differs from that of [8] by multiplication by 2.

Added in Proof. The question 2.4 has a negative answer in general, as pointed out by K. Oguiso.

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