74. *Mordell-Weil Lattices and Galois Representation. I*

By Tetsuji SHIODA
Department of Mathematics, Rikkyo University

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In the subsequent notes, we announce some results on the Mordell-Weil groups of elliptic curves over a function field or, equivalently, of elliptic surfaces. The key idea is to view such a Mordell-Weil group as a *lattice* with respect to the height pairing.

First, in the part I, we formulate the basic results on the Mordell-Weil groups from this viewpoint, which leads to some new results (Theorems 1.2 and 1.4).

Then, in the part II, we apply this theory to the case of rational elliptic surfaces, and we obtain the structure theorem for the Mordell-Weil lattices of such surfaces in the most interesting case, i.e., in the case of higher rank: the Mordell-Weil lattices of rank ≥ 6 are precisely E_8, E_7^*, E_6^* or D_6^* where $E_8, E_7, \cdots$ are the root lattices and $*$ indicates the dual lattices (Theorem 2.1). As a direct consequence, we can find *very effectively* the generators of such a Mordell-Weil group (Theorem 2.2). Next we make everything more explicit in terms of the Weierstrass form (Theorem 3.2). Another key idea is the use of the specialization map to an additive fibre (Lemma 3.3). In §4, we give some examples of the elliptic surfaces of Delsarte type.

In the part III, we discuss the Galois representations arising from the Mordell-Weil lattices. We can essentially answer the problem raised by Weil and Manin ([15, p. 558], [6, Ch. 4: 23. 13]).

1. The Mordell-Weil lattices. Let k be an algebraically closed field of arbitrary characteristic. Let $K=k(C)$ be the function field of a smooth projective curve C over k. Let E be an elliptic curve defined over K, given with a K-rational point O, and let $E(K)$ denote the group of K-rational points of E, with the origin O.

We consider the associated elliptic surface $f: S \to C$ (the Kodaira-Néron model of E/K). By this we mean the following: S is a smooth projective surface defined over k and f is a morphism of S onto C such that 1) the generic fibre is E and 2) no fibres contain an exceptional curve of the first kind (i.e. a smooth rational curve with self-intersection number -1). The existence and the uniqueness, up to an isomorphism, of the Kodaira-Néron model is well-known ([4], [7], [13]).

Now the global sections of $f: S \to C$ are in a natural one-to-one correspondence with the K-rational points of E. Thus we use the same notation $E(K)$ to denote the group of sections of f. For $P \in E(K)$, (P) will denote the prime divisor of S which is the image of the section $P: C \to S$.

We assume throughout that (*) f is *not smooth*, i.e., there is at least one singular fibre. Then $E(K)$ is finitely generated by the Mordell-Weil theorem, and it is called the Mordell-Weil group of the elliptic curve E/K, or of the elliptic surface $f: S \to C$.

We shall introduce the notion of Mordell-Weil lattices, i.e., the structure of positive-definite lattice on the Mordell-Weil group modulo torsion, by embedding the latter to the Néron-Severi group.

Let $NS(S)$ be the Néron-Severi group of S. It is defined as the group of divisors modulo algebraic equivalence, and known to be finitely generated. Under the assumption (*), it is torsion-free. The intersection number $(D_1 D_2)$ defines the structure of an integral lattice on $NS(S)$.

Let T be the sublattice of $NS(S)$ generated by the zero section and all the irreducible components of fibres. We call T the *trivial sublattice* of $NS(S)$. Let $L = T^{\perp}$ be the orthogonal complement of T in $NS(S)$, which we call the *essential sublattice* of $NS(S)$. Then L is a negative-definite even integral lattice; this is a consequence of the Hodge index theorem, the adjunction formula and the canonical bundle formula of an elliptic surface.

On the other hand, let $E(K)^0$ be the subgroup of finite index in the Mordell-Weil group $E(K)$ consisting of those sections which pass through the same irreducible component of every fibre as the zero section. With these notation, we can state the two fundamental theorems on the Mordell-Weil groups, of which the first one is well-known (cf. [9, §1]) while the second is new:

Theorem 1.1. *There is a natural isomorphism:*

$$E(K) \simeq NS(S)/T. \tag{1.1}$$

(The isomorphism is given by the map $P \to (P) \bmod T$, with the inverse map $D \to \mathrm{sum}(D|_E)$, the sum of the divisor $D|_E$ on E in the sense of the group law.)

Theorem 1.2. *There is a unique homomorphism* $\varphi: E(K) \to NS(S)_{\mathbf{Q}}$ *such that* $\varphi(P) \equiv (P) \bmod T_{\mathbf{Q}}$ *and* $\mathrm{Im}(\varphi) \perp T$. *The map* φ *fits in the commutative diagram:*

$$\begin{array}{ccccc} 0 \longrightarrow E(K)_{\mathrm{tor}} \longrightarrow & E(K) & \xrightarrow{\varphi} & L^* \\ & \cup & & \cup \\ & E(K)^0 & \simeq & L \end{array} \tag{1.2}$$

with exact rows, where L^* *is the dual lattice of* L.

Definition 1.3. For $P, P' \in E(K)$, let

$$\langle P, P' \rangle = -(\varphi(P) \cdot \varphi(P')). \tag{1.3}$$

This defines the structure of positive-definite lattices on $E(K)^0$ and $E(K)/E(K)_{\mathrm{tor}}$, which will be called the *narrow Mordell-Weil lattice* and the *Mordell-Weil lattice* of E/K or of $f: S \to C$. Note that the former is a positive-definite even integral lattice.

Theorem 1.4. *With the notation of Theorem* 1.2, *assume further that* $NS(S)$ *is unimodular. Then the map* φ *induces:*

$$(1.4)\qquad \begin{array}{ccc} E(K)/(\text{tor}) & \simeq & L^* \\ \cup & & \cup \\ E(K)^0 & \simeq & L. \end{array}$$

Moreover we have $[L^*: L] = \det L = (\det T)/|E(K)_{\text{tor}}|^2$.

Proposition. 1.5. *Let K' be a finite extension of K. Then*

$$(1.5)\qquad \langle P, P'\rangle_{E(K')} = [K': K]\langle P, P'\rangle_{E(K)}$$

for any $P, P' \in E(K)$.

The above results can be made more explicit in terms of the singular fibres of $f: S \to C$. Let F be any fixed fibre and let R be the set of the reducible fibres of f. For each $v \in R$, write

$$f^{-1}(v) = \Theta_{v,0} + \sum_{i\geq 1} \mu_{v,i}\Theta_{v,i} \qquad (\mu_{v,0}=1)$$

where $\Theta_{v,i}$ $(0 \leq i \leq m_v - 1)$ are the irreducible components, m_v being their number, such that $\Theta_{v,0}$ is the unique component of $f^{-1}(v)$ meeting the zero section. The trivial sublattice T of $NS(S)$ is the direct sum of $\langle (O), F\rangle$ and $T_v = \langle \Theta_{v,i}\ (i \geq 1)\rangle$ $(v \in R)$, and we have

$$(1.6)\qquad rk(T) = 2 + \sum_{v\in R}(m_v - 1),$$

$$(1.7)\qquad \det T = \prod_{v\in R} m_v^{(1)}, \qquad m_v^{(1)} = \det T_v = \#\{i \geq 0 \mid \mu_{v,i} = 1\}.$$

Let $\rho(S) = rk NS(S)$ and $r = rk E(K)$. Then Theorem 1.1 gives

$$(1.8)\qquad \rho(S) = r + 2 + \sum_{v\in R}(m_v - 1).$$

Next, the map φ is explicitly given by the formula:

$$(1.9)\quad \varphi(P) = (P) - (O) - ((PO) - (O^2))F - \sum_{v\in R}(\Theta_{v,1}, \cdots, \Theta_{v,m_v-1})A_v^{-1}\begin{pmatrix}(P\Theta_{v,1}) \\ \vdots \\ (P\Theta_{v,m_v-1})\end{pmatrix}$$

where (PO) is the abbreviation of $((P)(O))$ and A_v is the negative definite matrix $((\Theta_{v,i}\Theta_{v,j}))_{i,j\geq 1}$ of size $(m_v - 1)$. Hence the explicit form of the height pairing is:

$$(1.10)\qquad \langle P, P'\rangle = \chi + (PO) + (P'O) - (PP') - \sum_{v\in R} \text{contr}_v(P, P').$$

Here χ is the arithmetic genus of S ($\chi > 0$ under $(*)$) and the local contribution $\text{contr}_v(P, P')$ is zero if P or P' passes $\Theta_{v,0}$ and is equal to the (i, j)-entry of $(-A_v^{-1})$ if $(P\Theta_{v,i}) = (P'\Theta_{v,j}) = 1$ with $i, j \geq 1$. In particular,

$$(1.11)\qquad \langle P, P'\rangle = \chi + (PO) + (P'O) - (PP') \in \mathbf{Z} \qquad \text{if } P \text{ or } P' \in E(K)^0,$$

$$(1.12)\qquad \langle P, P\rangle = 2\chi + 2(PO) - \sum_{v\in R} \text{contr}_v(P)$$

where $\text{contr}_v(P) = (-A_v^{-1})_{i,i}$ if $(P\Theta_{v,i}) = 1$, $i \geq 1$, and $=0$ if $i = 0$. For any $P \neq O$, we have

$$(1{,}13)\qquad \langle P, P\rangle \geq 2\chi - \sum_{v\in R} \text{contr}_v(P),$$

and $P \in E(K)_{\text{tor}}$ if and only if the right hand side is zero.

Remark. The "height" pairing $\langle P, P'\rangle$ defined above is essentially the same as the canonical height due to Néron, Tate or Manin (cf. [5], [14]). Notice, however, that our definition gives more explicit formula (1.10). Also it coincides, in the complex case $k = \mathbf{C}$, with the pairing defined by Cox-Zucker [3], which is based on [9, §1] but which involves some transcendental argument. Thus the novelty in our approach is the idea to view the Mordell-Weil group as a lattice. It turns out to be very fruitful. Detailed exposition of the above results is in preparation [12].

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