29. *Hypoellipticity and Existence of Periodic Solutions on T^d*

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1. Introduction. In this paper we shall study the solvability and the regularity of linear and semilinear equations on T^d which are not necessarily elliptic. We know, by examples, that the regularity and the solvability of such operators are often expressed by diophantine conditions, such as a Siegel condition, etc. Hence it is interesting to seek operators for which necessary and sufficient conditions for the solvability or the regularity are equivalent to Siegel-type diophantine conditions.

Roughly speaking, such operators are characterized by (A. 1) which follows (cf. Remark 2. 4). Then we study them in connection with the global hypoellipticity and solvability on T^d, and the existence of periodic solutions for semilinear equations on T^d whose ratio of periods is not necessarily rational. In the former case, these operators clearly reveal the difference of the hypoellipticity and the global hypoellipticity which is still in question (cf. [1], [2]). In the latter case, the general theory does not work because of small denominators (cf. [4]). We can show the existence of solutions in this case.

§2. Notations and results. Let $T^d=R^d/(2\pi)Z^d$ be a d-dimensional torus. We denote the variables in T^d by $x=(x_1, \cdots, x_d)$. For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_d)\in N$, $N=\{0,1,2,\cdots\}$ we set $D^\alpha=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_d)^{\alpha_d}$. For $\beta=(\beta_1,\cdots,\beta_d)\in Z^d$ and $z=(z_1,\cdots,z_d)\in C^d$ we set $\langle\beta\rangle=1+|\beta|$, $|\beta|=|\beta_1|+\cdots+|\beta_d|$ and $\|z\|=(|z_1|^2+\cdots+|z_d|^2)^{1/2}$. A function $f(x)$ of $x\in R^d$ is identified with a function on the torus T^d if $f(x+2\pi\eta)=f(x)$ for all $x\in R^d$ and $\eta\in Z^d$. We denote the sets of distributions and infinitely differentiable functions on T^d, respectively by $\mathcal{D}'(T^d)$ and $C^\infty(T^d)$.

For $s\geq 1$, we define a Gevrey class $G^s(T^d)$ of order s by

$$G^s(T^d)=\{f=\sum_{\gamma\in Z^d} f_\gamma e^{i\gamma x}\in C^\infty(T^d)\,|\,\exists\varepsilon>0,\ \exists K>0 \text{ such that } |f_\gamma|\leq K\exp(-\varepsilon|\gamma|^{1/s}),\ \forall\gamma\in Z^d\}. \tag{2.1}$$

We remark that the above definition of a Gevrey class agrees with the usual one.

Let $m\geq 1$ be an integer, and let $p_m(\eta)$ be a polynomial of degree m, $p_m(\eta)=\sum_{|\alpha|=m} a_\alpha\eta^\alpha$ with $a_\alpha\in C$, and let $b(x,D)$ be a classical pseudo-differential operator on T^d of order $m-1$. We denote by $b(x,\eta)\in C^\infty(T^d\times R^d)$ the symbol of $b(x,D)$. We assume that $\langle\eta\rangle^{1-m}b(x,\eta)$ is uniformly in $G^s(T^d)$, namely we can take ε and K in (2.1) independent of $\eta\in Z^d$. We consider the following operator on T^d:

(2.2) $$P(D)\equiv p_m(D)+b(x, D).$$

We expand $b(x, \eta)$ into Fourier series, $b(x, \eta)=\sum_\gamma b_\gamma(\eta)e^{i\gamma x}$, and we define Γ_P as the smallest closed convex cone with apex at the origin which contains all γ such that $b_\gamma(\eta)\not\equiv 0$. If $b_\gamma(\eta)\equiv 0$ for all γ, then we set $\Gamma_P=\{0\}$. We assume

(A.1) Γ_P is a proper cone, that is, Γ_P contains no ray.

(A.2) For every $\xi\in \boldsymbol{R}^d$ such that $p_m(\xi)=0$, we have $\theta\cdot\nabla_\xi p_m(\xi)\neq 0$ for $\forall\theta\in\Gamma_P\backslash\{0\}$.

We say that P is G^s-globally hypoelliptic if $u\in\mathcal{D}'(\boldsymbol{T}^d)$ and $Pu\in G^s(\boldsymbol{T}^d)$ imply $u\in G^s(\boldsymbol{T}^d)$. The operators (2.2) are not hypoelliptic in general even if we assume (A.1) and (A.2). We set $p(\eta)=p_m(\eta)+b_0(\eta)$. Then we have

Theorem 2.1. *Suppose* (A.1) *and* (A.2). *Then, P is G^s-globally hypoelliptic if and only if*

(2.3) $$\lim_{\eta\to\infty,\eta\in \boldsymbol{Z}^d}|\eta|^{-1/s}\log|p(\eta)|=0.$$

Theorem 2.2 (Solvability). *Suppose* (A.1) *and* (A.2). *Then, the equation $Pu=f$ is uniquely solvable in $G^s(\boldsymbol{T}^d)$ if and only if, $p(\eta)\neq 0$ for all $\eta\in \boldsymbol{Z}^d$, and* (2.3) *is satisfied.*

Corollary 2.3 (cf. [1]). *Suppose that $b(x,\eta)\equiv 0$. Then P is G^s-globally hypoelliptic if and only if* (2.3) *is satisfied.*

Remark 2.4. We can replace (A.2) with a weaker condition. For example, if $d=2$ then we can replace it by $p_m(\xi)\neq 0$ for $\forall\xi\in\Gamma_P\backslash\{0\}$. Concerning this we refer [6]. On the other hand, (A.1) is sharp in general. In fact, the Mathieu operator $D_1^2+2\cos x_1$ on $\boldsymbol{T}^2$ is globally hypoelliptic.

Next we consider the existence of periodic solutions of semilinear equations on $\boldsymbol{T}^d$

(2.4) $$Pu\equiv Q(D)u+g(u)=f(x),\ Q(D)=\sum_{|\alpha|\le m}a_\alpha D^\alpha,\ g(u)=\sum_{j=2}^{l}g_ju^j$$

where $a_\alpha\in \boldsymbol{C}$, $l\geq 2$, $l\in \boldsymbol{Z}$ and where $f(x)\in C^\infty(\boldsymbol{T}^d)$ is a given function.

Let us expand $f(x)$ into Fourier series, $f(x)=\sum_\eta\hat{f}_\eta e^{i\eta x}$. We define the Fourier support, $\mathrm{supp}\hat{f}$ of f by $\mathrm{supp}\hat{f}=\{\eta;\eta\in \boldsymbol{Z}^d, \hat{f}_\eta\neq 0\}$. We set $q(\eta)=\sum_{|\alpha|\le m}a_\alpha\eta^\alpha$, and we denote by $q_m(\eta)$ the m-th homogeneous part of $q(\eta)$. We first consider the homogeneous case, $f=0$. Then there are l trivial solutions $u=u_0$ (constant) given by

(2.5) $$q(0)u_0+g(u_0)=0.$$

We are interested in the existence of nontrivial (non-constant) solutions near trivial solutions. We can show that

Theorem 2.5. *Let Γ be any convex closed proper cone with apex at the origin such that $q_m(\eta)\neq 0$ for $\forall\eta\in\Gamma\backslash\{0\}$. Let u_0 be any solution of* (2.5). *Then* Eq. (2.4) *with $f=0$ has a nontrivial smooth solution u such that* $\mathrm{supp}\,\hat{u}\subset\Gamma$, $u_0=(2\pi)^{-d}\int u(x)dx$ *if and only if $q(\eta)+g'(u_0)=0$ for some $\eta\in\Gamma\cap \boldsymbol{Z}^d$.*

Suppose $u\equiv u_0$ be a trivial solution of (2.5) such that there is no nontrivial solution v near u_0 in the above sense. Then we have

Theorem 2.6. *Let u_0 be as above. Then, there exists an $\varepsilon>0$ such that, for any $f\in C^\infty(\boldsymbol{T}^d)$ satisfying $\|f\|_2\le\varepsilon$,* $\operatorname{supp}\hat{f}\subset\tilde{\Gamma}$ Eq. (2.4) *has a solution u such that* $\operatorname{supp}\hat{u}\subset\tilde{\Gamma}$. *Moreover, it is unique near u_0 in the set of functions with Fourier supports contained in $\tilde{\Gamma}$. Here $\|\cdot\|_2$ is given by $\|f\|_2=\sum_\eta\langle\eta\rangle^2|f_\eta|$, $f=\sum_\eta f_\eta e^{i\eta x}$.*

As applications of the above theorems we give existence theorems for equations whose ratio of periods is not necessarily rational. Let Γ be any closed convex proper cone with apex at the origin such that $q_m(\eta)\ne 0$ for $^\forall\eta\in\Gamma\backslash\{0\}$. Then we have

Corollary 2.7. *Suppose $g(u)=u^2$. Then* (2.4) *with $f=0$ has a nontrivial smooth solution u such that* supp $\hat{u}\subset\Gamma$ *if and only if $q(\eta)(q(\eta)-2q(0))=0$ for some $\eta\in\Gamma\cap\boldsymbol{Z}^d$.*

Corollary 2.8. *Suppose $g=u^2$. Then, there exists an $\varepsilon>0$ such that, for any $f\in C^\infty(\boldsymbol{T}^d)$ satisfying* $\operatorname{supp}\hat{f}\subset\tilde{\Gamma}$ *and*

$$\|f\|_2\le\varepsilon\,;\ f_0\equiv(2\pi)^{-d}\int f\,dx\ne q(\eta)^2/4-q(0)q(\eta)/2,\ ^\forall\eta\in\tilde{\Gamma}\backslash\{0\},$$

Eq. (2.4) *has two solutions $u\in C^\infty(\boldsymbol{T}^d)$ so that* $\operatorname{supp}\hat{u}\subset\tilde{\Gamma}$ *in a small neighborhood of two trivial solutions, respectively.*

References

[1] S. Greenfield and N. Wallach: Global hypoellipticity and Liouville numbers. Proc. Amer. Math. Soc., **31**, 112–114 (1972).

[2] C. Siegel: Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung. Nachr. Akad. Wiss. Göttingen, pp. 21–30 (1952).

[3] K. Taira: Le principe du maximum et l'hypoellipticité globale. Séminaire Bony-Sjöstrand-Meyer 1984–1985, n° 1.

[4] K. Tanaka: Infinitely many periodic solutions for the equation. $u_{tt}-u_{xx}\pm|u|^{s-1}u=f(x,t)$. Comm. in P.D.E., **10**, 1317–1345 (1985).

[5] M. Yoshino: An application of generalized implicit function theorem to Goursat problems for nonlinear Leray-Volevich systems. J. Diff. Eqs., **57**, 44–69 (1985).

[6] ——: The diophantine nature for the convergence of formal solutions. Tôhoku J. Math., **38**, 625–641 (1986).