# 20. *Tightness Property for Symmetric Diffusion Processes*

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**§1. Introduction.** Let $\mathcal{E}^n$ be a sequence of closable symmetric forms on $L^2(\mathrm{R}^d, m_n)$ with symmetric non-negative definite (in $i, j$) measurable coefficients $a^n_{i,j}$:

$$\mathcal{E}^n(f, g)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathrm{R}^d} a^n_{ij}(x)\frac{\partial f}{\partial x_i}(x)\frac{\partial g}{\partial x_j}(x)dm_n$$

$$\mathcal{D}[\mathcal{E}^n]=C_0^\infty(\mathrm{R}^d)$$

where $m_n$ are everywhere dense positive Radon measures and $C_0^\infty(\mathrm{R}^d)$ is the space of infinitely differentiable functions with compact support. We assume that there exists a positive constant $c$ such that

$$\sup_n \sum_{i,j=1}^{d} a^n_{ij}(x)\xi_i\xi_j \leqq c|\xi|^2$$

for all $x$ and $\xi\in\mathrm{R}^d$. Set $\mathcal{E}^n_1(f, g)=\mathcal{E}^n(f, g)+(f, g)_{m_n}$ and denote the $\mathcal{E}^n_1$-closure of $C_0^\infty$ by $\mathcal{F}^n$. Then we have a sequence of regular Dirichlet spaces $(\mathcal{E}^n, \mathcal{F}^n)$ on $L^2(\mathrm{R}^d, m_n)$ and symmetric diffusion processes $\mathrm{M}^n=(P^n_x, X_t)$ associated with $(\mathcal{E}^n, \mathcal{F}^n)$ (see [3]).

For the probability measure $\mu_n$ on $\mathrm{R}^d$, we define the probability measure $P^n_{\mu_n}$ on $C([0, \infty))$ as $P^n_{\mu_n}(\cdot)=\int P^n_x(\cdot)d\mu_n$, where $C([0, \infty))$ is the space of all continuous functions from $[0, \infty)$ into $\mathrm{R}^d$. We are concerned with the problem of finding conditions for a sequence $\{P^n_{\mu_n}\}$ to be tight.

**§2. Statement of theorem.** We consider the following conditions.

Condition 1. Diffusion processes $\mathrm{M}^n$ are conservative.

Condition 2. i) $\sup_n m_n(K)<\infty$ for any compact set $K$

ii) $\mu_n=\phi_n dm_n$ and $\sup_n \|\phi_n\|_\infty<\infty$

iii) $\{\mu_n\}$ is tight

Condition 3. For any $T>0$ and $R>0$

$$\sup_n \sum_{k=0}^{\infty} m_n(T_{R+k})l^{1/2}\left(\frac{k}{\sqrt{dcT}}\right)<\infty$$

where

$$T_p=\{x\in\mathrm{R}^d\,;\,p\leqq|x|<p+1\}\quad\text{and}\quad l(a)=\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-u^2/2}du.$$

Then, we have

**Theorem.** *Under Conditions* 1, 2 *and* 3, *the sequence of probability measure* $\{P^n_{\mu_n}\}$ *is tight.*

**Remark 1.** Under Condition 1 and Condition 2-i), ii), Lyons-Zheng [4] have proved that $\{P^n_{\mu_n}\}$ is tight as a sequence of probability measures on $D([0, \infty))$, the space of all right continuous functions with left-hand limits. But they assumed that $D([0, \infty))$ is endowed with pseudo-path topology weaker than Skorohod's one.

**Remark 2.** Consider the case that the measure $m_n$ is absolutely continuous with respect to Lebesgue measure, say $m_n = \Psi_n dx$, and let

$$\bar{\Psi}_n(r) = \int_{S^{d-1}} \Psi_n(r, \sigma) d\sigma, \qquad \text{for } r > 0$$

where $d\sigma$ is the uniform measure on $S^{d-1}$. If there exists a positive constant $\varepsilon$ such that

$$\sup_n \bar{\Psi}_n(r) < e^{r^{2-\varepsilon}},$$

then Condition 3 is fulfilled.

**Remark 3.** In Albeverio-Høegh-Krohn-Streit [1] and Albeverio-Kusuoka-Streit [2], the convergence of finite dimensional distribution of $P^n_{\mu_n}$ was investigated in case that Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ are energy forms. In some of their examples we can check our conditions and conclude the weak convergence.

**§ 3. Outline of the proof of theorem.** Set $P^n_{m_n} = \int P^n_x \, dm_n$. Then, $P^n_{m_n}$ is a $\sigma$-finite measure on $C([0, \infty))$.

**Lemma 1.** *For Borel sets $A$ and $B \subset \mathbf{R}^d$*

$$P^n_{m_n}[X_0 \in A, X_T \in B] \leqq 4d(m_n(A) + m_n(B)) \cdot l\left(\frac{\rho(A, B)}{\sqrt{cT}}\right)$$

*where*

$$\rho(A, B) = \inf \{\rho(x, y) \mid x \in A, y \in B\} (\rho(x, y) = \max_{1 \leqq i \leqq d} |x^i - y^i|).$$

By using this lemma, we get the following inequality.

**Lemma 2.**

$$P^n_{\mu_n}[\sup_{\substack{0 \leqq s, t \leqq T \\ |t-s| \leqq h}} \rho(X_t, X_s) > \delta] \leqq d \cdot P^w[\sup_{\substack{0 \leqq s, t \leqq cT \\ |t-s| \leqq ch}} |B_t - B_s| > \delta]^{1/2}$$
$$\cdot \Big\{ \mu_n(B_R) + \|\phi_n\|_\infty (m_n(B_R) + 2\sqrt{d}(1 + m_n(B_R))^{1/2})$$
$$\cdot \sum_{k=0}^{\infty} (1 + m_n(T_{R+k})) l^{1/2}\left(\frac{k}{\sqrt{dcT}}\right)\Big\} + \mu_n(B^c_R),$$

*where $B_R = \{x \in \mathbf{R}^d ; |x| \leqq R\}$ and $P^w$ is 1-dimensional Wiener measure.*

By Lemma 2, Condition 2 and Condition 3, it holds that for any $T > 0$

$$\lim_{h \downarrow 0} \sup_n P^n_{\mu_n}[\sup_{\substack{0 \leqq s, t \leqq T \\ |t-s| \leqq h}} |X_t - X_s| > \delta] = 0$$

and we obtain the theorem.

For the proof of Lemma 1 and Lemma 2, we use the fact that the functional $X^i_t - X^i_0$ $(0 \leqq t \leqq T, 1 \leqq i \leqq d)$ can be written as the sum of a $(P_m, \mathcal{F}_t)$-local martingale and a $(P_m, \mathcal{G}_t)$-one. Here $\mathcal{F}_t$ and $\mathcal{G}_t$ are $\sigma$-field generated by $\{X_t ; 0 \leqq s \leqq t\}$ and $\{X_{T-s} ; 0 \leqq s \leqq t\}$ respectively (see [4]).

## References

[1] S. Albeverio, R. Høegh-Krohn, and L. Streit: Regularization of Hamiltonians and processes. J. Math. Phys., **21**, 1636–1642 (1980).

[2] S. Albeverio, S. Kusuoka, and L. Streit: Convergence of Dirichlet forms and associated Schrödinger operators. J. Funct. Anal., **68**, 130–148 (1986).

[3] M. Fukushima: Dirichlet Forms and Markov Processes. Kodansha and North Holland (1980).

[4] T. Lyons and W. A. Zheng: Crossing estimate for canonical process on a Dirichlet space and a tightness result (preprint).