17. *Single-point Blow-up for Semilinear Parabolic Equations in Some Non-radial Domains*

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§ 0. **Introduction.** In this note, we consider

$$\text{(E)}\qquad \begin{cases} u_t=\Delta u+f(u), & (t,x)\in(0,T)\times\Omega,\\ u=0, & (t,x)\in(0,T)\times\partial\Omega,\\ u(0,x)=u_0(x), & x\in\bar{\Omega}. \end{cases}$$

Here $\Omega\subset\boldsymbol{R}^N$ $(N\geqq 2)$ is a bounded domain with smooth boundary and the initial value $u_0=u_0(x)\geqq 0$ is sufficiently smooth, say, $u_0\in C^1(\bar{\Omega})\cap C_0(\bar{\Omega})$. The nonlinear term $f(u)$ satisfies

(0.1) $\qquad f\in C^2(0,\infty)\cap C[0,\infty),\quad f(s)>0 \qquad \text{for } s>0.$

Let $u=u(t,x)$ be the classical solution of (E). Its existence time T is defined by

(0.2) $\qquad T=\sup\{\tau>0\,|\,u(t,x) \text{ is bounded in } [0,\tau]\times\Omega\}.$

It is well known that for a large class of f and initial value u_0, the solution $u(t,x)$ may blow up, i.e., $T<+\infty$ and

(0.3) $\qquad \overline{\lim_{t\uparrow T}}\, \|u(t,\cdot)\|_{L^\infty(\Omega)}=+\infty.$

In this case we say that $u=u(t,x)$ is a *blow-up solution*, and T is the *blow-up time* (see, for instance [3], [4]).

Here, we consider the blow-up points in some non-radial domains and will give some single-point blow-up results under a weaker hypothesis than the radial symmetry or convexity for Ω.

Definition. The *blow-up set*, or the *set of blow-up points* of $u=u(t,x)$ is defined as

$$S=\{x\in\bar{\Omega}\,|\,\text{there is a sequence } (t_n,x_n) \text{ in } (0,T)\times\Omega \text{ such that } t_n\uparrow T,\ x_n\to x \text{ and } u(t_n,x_n)\to\infty \text{ as } n\to+\infty\},$$

and each point $x\in S$ is called a *blow-up point* of $u(t,x)$.

By the definition, we can see that S is a closed set. The standing assumption throughout this note is that $f(\cdot)$ and u_0 is such that the solution blows up. For f we assume the following condition.

(F) There exists a function $F=F(u)$ such that

(i) $F(s)>0,\quad F'(s)\geqq 0 \quad\text{and}\quad F''(s)\geqq 0 \qquad \text{for } s>0;$

(ii) $\displaystyle\int_1^\infty \frac{ds}{F(s)}<+\infty;$

(iii) there is a constant $\sigma>0$ such that

$$f'(s)F(s)-f(s)F'(s)\geqq\sigma F(s)F'(s) \qquad \text{for } s>0.$$

This condition is originally introduced in [6]. It can be seen that

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$f=u^p$ ($p>1$), $f=\lambda e^{\mu u}$ ($\lambda>0$, $\mu>0$) or $f=au^p+bu^q$ ($a,b>0$, $p,q>1$) satisfies (F).

With the concepts to be defined later in § 1, our main result reads as follows.

Main Theorem. *Let $\{\gamma_j\}_{j=1}^N$ be a set of independent unit vectors and $\{T_j\}_{j=1}^N$ be hyperplanes defined by $T_j=\{x\in \boldsymbol{R}^N \mid x\cdot\gamma_j=c_j\}$ for $c_j\in \boldsymbol{R}^1$ such that $T_j\cap\Omega\neq\emptyset$, $j=1,\cdots,N$. Suppose that Ω has weak Gidas, B., W.-M. Ni, and L. Nirenberg (GNN) symmetry for each T_j and f satisfies (F). If for each T_j, $u_0(x)$ is symmetric and weakly GNN decreasing along $\pm\gamma_j$ ($j=1,\cdots,N$), then the solution blows up only at a single point. Actually, the blow-up set is nothing but $S=\bigcap_{j=1}^N T_j$.*

We would like to mention that the results can also be extended to the cases concerning the Neumann or Robin boundary condition, and for some unbounded domains such as $\boldsymbol{R}^N$, or domains with corner points in $\partial\Omega$.

§ 1. Definitions of GNN properties. We recall some concepts introduced by B. Gidas, W.-M. Ni and L. Nirenberg in [7].

Let $\gamma\in\boldsymbol{R}^N$ be a unit vector and T_λ be the hyperplane defined by $T_\lambda=\{x\in\boldsymbol{R}^N \mid x\cdot\gamma=\lambda\}$ for a real number λ. Since Ω is bounded, if $|\lambda|>\sup\{|x| \mid x\in\Omega\}$ then $T_\lambda\cap\Omega=\emptyset$. Put

(1.1) $$\delta^*=\sup\{\delta \mid T_\delta\cap\Omega\neq\emptyset\},\qquad \delta_*=\inf\{\delta \mid T_\delta\cap\Omega\neq\emptyset\}.$$

Then $-\infty<\delta_*<\delta^*<+\infty$ and $T_\lambda\cap\Omega\neq\emptyset$ if $\delta_*<\lambda<\delta^*$. For T_λ and a point $x\in\boldsymbol{R}^N$ with $x\notin T_\lambda$, the *reflection* (or *symmetric point*) of x for T_λ is a point $x'\in\boldsymbol{R}^N$ such that the line segment connecting x and x' is orthogonal to T_λ with $\operatorname{dist}(x,T_\lambda)=\operatorname{dist}(x',T_\lambda)$, where $\operatorname{dist}(\cdot,\cdot)$ indicates the Euclidean distance.

Definition 1.1 (Weak GNN property). Let $\lambda\in(\delta_*,\delta^*)$, $\delta\in(\lambda,\delta^*)$ and put $G=\{x\in\Omega \mid x\cdot\gamma>\lambda\}$, $G(\delta)=\{x\in\Omega \mid x\cdot\gamma>\delta\}$. We say that the domain Ω has weak GNN property for T_λ along the direction γ or G is a weakly GNN-type subdomain of Ω, if for each δ in (λ,δ^*), the reflection set $G'(\delta)$ of $G(\delta)$ for T_δ lies in Ω, where

$$G'(\delta)=\{x'\in\boldsymbol{R}^N \mid x' \text{ is the reflection for } T_\delta \text{ of } x, x\in G(\delta)\}.$$

Definition 1.2 (Strong GNN property). We say that the domain Ω has strong GNN property for T_λ along γ or G is a strongly GNN-type subdomain of Ω, if Ω has weak GNN property for T_λ along γ or equivalently G is a weakly GNN-type subdomain of Ω, and T_δ is not orthogonal to $\partial\Omega$ for each $\delta\in(\lambda,\delta^*)$.

Definition 1.3 (Local GNN property). Let G be a connected component of $\{x\in\Omega \mid x\cdot\gamma>\lambda\}$. If G satisfies the assumption of Definition 1.1 (respectively, Definition 1.2) then G is called a weakly (resp. strongly) GNN-type component of Ω for T_λ along γ.

Definition 1.4 (GNN symmetry). Let $\lambda\in(\delta_*,\delta^*)$. We say that Ω has strong (respectively, weak) GNN symmetry, or Ω is strongly (resp. weakly) GNN symmetric for T_λ, if Ω has strong (resp. weak) GNN property for T_λ both along γ and $-\gamma$.

It is obvious that the strong GNN symmetry implies the weak one.

Definition 1.5 (GNN decreasing property of a function). Let $g(x) \in C(\Omega)$ and G be a weakly GNN-type subdomain, or component, of Ω for T_λ along γ. We say that $g(x)$ is strongly GNN decreasing (resp. weakly GNN decreasing) along γ, if for each $\delta \in (\lambda, \delta^*)$,

(1.2) $$g(x)<g(x') \quad (\text{resp. } g(x) \leqq g(x')), \quad x \in G(\delta)$$

holds true, where x' is the reflection of x for T_δ.

§2. Fundamental lemmas concerning blow-up sets. For each two nonzero vectors y, z in $\boldsymbol{R}^N$, let $\theta=\langle y, z\rangle \in [0, \pi]$ be the angle between them. As an improvement to the general case of a result in [2], we have the following fundamental lemma.

Lemma 2.1. *Let u be a positive solution of (E), Q be an open subset of Ω and ν be a nonzero vector in $\boldsymbol{R}^N$. Suppose that f satisfies (F). If there is a constant σ in $(0, \pi/2)$ and a time $\tau \in (0, T)$ such that the angle $\theta=\langle \nu, \nabla u\rangle$ is confined in $[0, \sigma]$ or $[\pi-\sigma, \pi]$ for all $(t, x) \in [\tau, T) \times Q$, then there is no blow-up point in Q.*

This lemma can be proved by the method of [6] of introducing some auxiliary function $J(t, x)$. However, we take some different form of their J, and consequently our argument does not depend on the boundary condition of u.

The following lemma also follows from the argument of [6] of reflecting some portion of Ω.

Lemma 2.2. *Let G be a strongly GNN-type component of Ω. Suppose that f satisfies (F) and $u_0(x)$ is weakly GNN decreasing and non-constant in G. Then there is no blow-up point in G.*

With Lemma 2.2, we can show a single-point blow-up result for strongly GNN symmetric domains.

§3. Outline of proof of main theorem. From Lemma 2.1, we can derive the next.

Proposition 3.1. *Under the assumption of Main Theorem, the blow-up set is located in $\cup_{j=1}^N T_j$.*

Outline of the proof for main theorem. For simplicity, we only deal with the case $T_j=\{x_j=0\}$. There is no blow-up point in $\Omega \backslash (\cup_{j=1}^N T_j)$ by Proposition 3.1. For each j, we can see from GNN property and the smoothness of $\partial\Omega$, that there exists a constant $\lambda \in (\delta_*, \delta^*)$ such that $G=\{x \in \Omega \mid x_j>\lambda\}$ is a strongly GNN-type subdomain of Ω for $T_\lambda=\{x \in \boldsymbol{R}^N \mid x_j=\lambda\}$ along e_j, the positive direction of x_j. Thus, there is no blow-up point in a neighborhood of P_j, the intersection point of positive x_j-axis and $\partial\Omega$, by means of Lemma 2.2. Similar argument can also be made in the negative direction of x_j-axis for $j=1, \cdots, N$. Hence there is a positive number $\varepsilon>0$ such that there is no blow-up point in the domain $Q_\varepsilon=\{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega)<\varepsilon\}$, namely, S is compact and $S \subset \Omega_\varepsilon \cap (\cup_{j=1}^N T_j)$, where $\Omega_\varepsilon=\Omega \backslash \overline{Q}_\varepsilon$.

Therefore, we can take a closed surface Γ in $Q_\varepsilon \cup \partial\Omega$ such that the subdomain Ω_0 enclosed by the boundary Γ is simply connected and has strong

GNN symmetry for T_j, $j=1, \cdots, N$, with $S \subset \Omega_0$. Let $\tau \in (0, T)$, then $u(\tau, x)$ is strongly GNN decreasing for the corresponding hyperplanes and vectors. Noting that

$$\sup \{f(u(t, x)) \mid (t, x) \in [0, T) \times \Gamma\} < +\infty$$

$\sup \{u_{x_j} \mid (t, x) \in [\tau, T) \times \bar{\Omega},\ x_j > \delta\} < 0$ for each small $\delta > 0$ $(1 \leqq j \leqq N)$, we can easily get the conclusion by a similar argument to the proof of main theorem for strongly GNN symmetric domains. Q.E.D.

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