12. *Asymptotic Behavior for Weak Solutions of a Porous Media Equation*[†]

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1. Introduction. This work concerns the asymptotic behavior for $t\to\infty$ of the solution to the initial-boundary value problem

(1) $$u_t=\{r(u)u_x+b(t,u)\}_x,\qquad (x,t)\in(0,1)\times(0,\infty)$$

(2) $$u(0,t)=\psi_0(t),\quad u(1,t)=\psi_1(t),\quad t\in(0,\infty)$$

(3) $$u(x,0)=u_0(x),\qquad x\in(0,1).$$

The functions $r(u)$, $b(t,u)$, $\psi_i(t)$ $(i=0,1)$ and $u_0(x)$ are all real, bounded and sufficiently smooth. Moreover, we require

(A1) $r(u)>0$ in $u\in(u_*,u^*)$, $r(u_*)=r(u^*)=0$;

(A2) $|b_u(t,u)|\leq C\sqrt{r(u)}$;

(A3) $u_*\leq\psi_i(t)$, $u_0(t)\leq u^*$, $|\psi_i'(t)|\leq C$;

(A4) $\psi_0(0)=u_0(0)$, $\psi_1(0)=u_0(1)$ (compatibility condition);

(A5) $|\psi_i(t)-\varphi_i|\to 0$, $\sup_u|b(t,u)-a(u)|\to 0$ as $t\to\infty$;

(A6) $|a'(u)|\leq Cr(u)$.

With these conditions (1) describes a diffusion of immiscible fluids in porous media. b representing effects of both the flow and gravity terms, in general, depends not only on u but also on t. If the flow term is negligible, it can be given independently on t. In such a case, the asymptotic behavior of solutions has been studied in [5] for a fast diffusion arising in the filtration in a partially saturated porous media. Our purpose of this note is to extend their method to obtain a similar result to the above mentioned problem.

2. Results. In the following we consider the weak solution of (1)–(3) which satisfies: i) $u(x,t)\in C(\bar{Q})$, $Q=(0,1)\times(0,\infty)$, $u_*\leq u(x,t)\leq u^*$ for all $(x,t)\in\bar{Q}$ and $R(u(x,t))_x\equiv\int_{u_*}^{u(x,t)}r(\sigma)d\sigma\in L^2_{\rm loc}(\bar{Q})$; ii) $u(i,t)=\psi_i(t)$ $(i=0,1)$, $t\geq 0$; iii) For any $T>0$ and $g(x,t)\in C^1(\bar{Q})$ satisfying $g(0,t)=g(1,t)=0$,

(4) $$\int_0^T\int_0^1[\{R(u(x,t)_x+b(t,u(x,t))\}g_x-u(x,t)g_t]dxdt =\int_0^1 u_0(x)g(x,0)dx-\int_0^1 u(x,T)g(x,T)dx.$$

The following two theorems are already known. (See [1], [4], where are treated the case $b=b(u)$. The dependence on t of b is inessential for the proof of these theorems.)

Theorem 1. (i) *Assume* (A1), (A3) *and* (A4). *Then there exists a*

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weak solution of (1)–(3). (ii) *Assume further* (A2). *Then the weak solutions of* (1)–(3) *are unique.*

Theorem 2. *Assume* (A1) *and* (A2). *Let* $u_j(x,t)$, $j=1,2$, *be two weak solutions of* (1)–(3) *with respective data* $\{\psi_{0j}(t), \psi_{1j}(t), u_{0j}(x)\}$ *satisfying* (A3) *and* (A4). *Suppose that* $\psi_{i1}(t)\leq\psi_{i2}(t)$ $(i=0,1)$ *for all* $t\geq 0$ *and* $u_{01}(x)\leq u_{02}(x)$ *for all* $0\leq x\leq 1$. *Then we have* $u_1(x,t)\leq u_2(x,t)$ *for all* $(x,t)\in\bar{Q}$.

The stationary problem associated with (1)–(3) is given by

$$(5)\qquad \{r(v)v'+a(v)\}'=0\ (0\leq x\leq 1),\qquad v(i)=\varphi_i\ (i=0,1).$$

We define the weak solution to satisfy: i) $v(x)\in C([0,1])$, $u_*\leq v(x)\leq u^*$ for all $0\leq x\leq 1$ and $R(v(x))'\in L^2((0,1))$; ii) $v(i)=\varphi_i$ $(i=0,1)$; iii) $\int_0^1\{R(v(x))'+a(v(x))\}f'(x)dx=0$ for any $f(x)\in C^1([0,1])$ satisfying $f(0)=f(1)=0$.

Now our main result is the following

Theorem 3. *Assume* (A1)–(A6). *Then as* $t\to\infty$ *the weak solution* $u(x,t)$ *of* (1)–(3) *converges uniformly in* $x\in[0,1]$ *to the weak solution* $v(x)$ *of* (5).

3. Lemmas. We prepare several lemmas under (A1)–(A6).

Lemma 4. *There exists a* $C>0$ *such that for any* $T>1$,

$$\int_T^{T+1}\int_0^1\{R(u(x,t))_t\}^2\,dxdt+\int_0^1\{R(u(x,T))_x\}^2\,dx\leq C.$$

Proof. Our weak solution of (1)–(3) is the limit of a uniform convergent sequence of approximate classical solutions. So, we can assume u being a classical solution. Let $\psi(x,t)=(1-x)\psi_0(t)+x\psi_1(t)$. Multiply both sides of (1) by $u(x,t)-\psi(x,t)$ $[R(u(x,t))_t-R(\psi(x,t))_t]$, and integrate by parts on $(0,1)\times(T-1,T+1)$ $[(0,1)\times(T',T+1)$ for a suitable $T-1\leq T'\leq T]$. Then we can easily obtain the above estimate. □

As a corollary of the above lemma we have

Lemma 5. *There exists a* $C>0$ *such that*

$$|R(u(x,t))-R(u(y,s))|\leq C\{|x-y|^{1/2}+|t-s|^{1/4}\}$$

for any $0\leq x,y\leq 1$ *and* $t,s\geq 1$ *such that* $|t-s|$ *is sufficiently small.*

Lemma 6. *The weak solutions of* (5) *are unique.*

Proof. We rewrite (5) as the equation of $w=R(v)$. Then by use of (A6) with $u=v$, we can show the uniqueness result (see e.g., [3], chapter 9). □

By use of the comparison Theorem 2 and the above Lemmas 4–6, we can follow the argument of [5] to obtain

Lemma 7. *Let* $\psi_{i-}(t)$ $[\psi_{i+}(t)]$ $(i=0,1)$ *be monotone increasing* [*decreasing*] *smooth functions such that*

$$u_*\leq\psi_{i-}(t)\leq\psi_i(t)\leq\psi_{i+}(t)\leq u^*,\quad |\psi'_{i\pm}(t)|\leq C\quad \textit{for all } t\geq 0,$$
$$\psi_{i-}(0)=u_*,\quad \psi_{i+}(0)=u^*\quad \textit{and}\quad \lim_{t\to\infty}\psi_{i\pm}(t)=\varphi_i.$$

Let $u_-(x,t)$ $[u_+(x,t)]$ *be the solution of* (1)–(3) *with* $b(t,u)$, $\psi_i(t)$ *and* $u_0(x)$ *replaced respectively by* $a(u)$, $\psi_{i-}(t)$ $[\psi_{i+}(t)]$ *and* $u_*[u^*]$. *Then as* $t\to\infty$, $u_\pm(x,t)\to v(x)$ *uniformly in* $x\in[0,1]$, *where* $v(x)$ *is the weak solution of* (5).

The following lemma also follows from the comparison theorem.

Lemma 8. *Let σ be an arbitrary fixed positive number. Let $u_{\sigma-}(x,t)$ $[u_{\sigma+}(x,t)]$ be the solution of* (1) *with $b(t,u)$, $\psi_i(t)$ and $u_0(x)$ replaced respectively by $b(t+\sigma,u)$, $\psi_{i-}(t)$ $[\psi_{i+}(t)]$ and $u_*[u^*]$. Then we have $u_{\sigma-}(x,t)\leq u(x,t+\sigma)\leq u_{\sigma+}(x,t)$ for any $(x,t)\in\overline{Q}$.*

Lemma 9. *There exists a $C>0$ such that for any $\sigma, T>0$,*

$$\int_T^{T+1}\int_0^1 |R(u_\pm(x,t))-R(u_{\sigma\pm}(x,t))|^2\,dxdt \leq Ce^{CT}\sup_{t\geq 0,u}|b(t+\sigma,u)-a(u)|.$$

Proof (cf., [7]). Put $w=u_\pm-u_{\sigma\pm}$. Then for any test function $g(x,t)$ such that $g(x,T+1)=0$, we have noting (4),

$$\int_0^{T+1}\int_0^1 w\{g_t+\tilde{R}(x,t)g_{xx}-\tilde{a}(x,t)g_x\}dxdt + \int_0^{T+1}\int_0^1 \{b(t+\sigma,u_{\sigma\pm})-a(u_{\sigma\pm})\}g_x dxdt=0, \tag{6}$$

where for each function $h(u)$, $\tilde{h}(x,t)$ is defined by

$$\tilde{h}(x,t)=\frac{h(u_\pm(x,t))-h(u_{\sigma\pm}(x,t))}{u_\pm(x,t)-u_{\sigma\pm}(x,t)}.$$

Consider the following backward initial-boundary value problem in $Q_{T+1}=(0,1)\times(0,T+1)$:

$$\begin{cases} g_t+\{\tilde{R}(x,t)+1/n\}g_{xx}-\tilde{a}(x,t)g_x=\tilde{R}(x,t)w, & (x,t)\in Q_{T+1} \\ g(0,t)=g(1,t)=0, & t\in(0,T+1) \\ g(x,T+1)=0, & x\in(0,1). \end{cases} \tag{7}$$

This is solvable in the Sobolev space $W_q^{2,1}(Q_{T+1})$, $q>1$, with norm

$$\|f\|_{q,Q_{T+1}}^{(2)}=\sum_{j=0}^{2}\|\partial_x^j f\|_{q,Q_{T+1}}+\|\partial_t f\|_{q,Q_{T+1}},$$

where $\|\ \|_{q,Q_{T+1}}$ is the usual L^q-norm in Q_{T+1} ([6], chapter IV). Let $g_n(x,t)$ be the solution of (7). Multiply by $g_{n,xx}$ on the both sides of the differential equation and integrate by parts on Q_{T+1}. Then, applying the Gronwall inequality, we easily have for any $0\leq t\leq T+1$,

$$\int_0^1 g_{n,x}^2(x,t)dx+\int_0^{T+1}\int_0^1(\tilde{R}+1/n)g_{n,xx}^2\,dxdt\leq Ce^{CT},$$

where $C>0$ is independent of n and T. In (6) we put $g=g_n$ and let $n\to\infty$. Then using this inequality, we conclude

$$\int_0^{T+1}\int_0^1 w^2\tilde{R}\,dxdt\leq Ce^{CT}\sup_{t\geq 0,u}|b(t+\sigma,u)-a(u)|.$$

Since $|R(u_\pm)-R(u_{\sigma\pm})|^2\leq(w\tilde{R})^2\leq Cw^2\tilde{R}$, this proves the lemma. □

4. Proof of Theorem 3. Noting (A5), we see that there exists a monotone decreasing continuous function $\varepsilon(\sigma)>0$ such that $\varepsilon(\sigma)\to 0$ as $\sigma\to\infty$ and $\sup_{t\geq 0,u}|b(t+\sigma,u)-a(u)|\leq\varepsilon(\sigma)$. With this $\varepsilon(\sigma)$ we can say that for any sufficiently large τ there exists a $\sigma(\tau)\leq\tau$ such that $e^{C(\tau-\sigma(\tau))}=\varepsilon(\sigma(\tau))^{-1/2}$. As is easily seen, $\sigma(\tau)\to\infty$ and $\tau-\sigma(\tau)\to\infty$ as $\tau\to\infty$. We put $T=T(\tau)=\tau-\sigma(\tau)$ and $\sigma=\sigma(\tau)$ in Lemma 9. Then

$$\int_{T(\tau)}^{T(\tau)+1}\int_0^1|R(u_\pm(x,t))-R(u_{\sigma(\tau)\pm}(x,t))|^2\,dxdt \leq Ce^{CT(\tau)}\sup_{t\geq 0,u}|b(t+\sigma(\tau),u)-a(u)|\leq C\varepsilon(\sigma(\tau))^{1/2}\to 0 \tag{8}$$

as $\tau \to \infty$. On the other hand, the set of functions

(9) $$\{R(u_{\pm}(x, s+T(\tau))) - R(u_{\sigma(\tau)\pm}(x, s+T(\tau))) \,;\, \tau \geq 1\}$$

is uniformly bounded and equi-continuous in $(x, s) \in [0,1] \times [0,1]$ (Lemma 5). Combining the Ascoli-Arzela theorem and (8), we see that as $\tau \to \infty$, $R(u_{\pm}(x, s+T(\tau))) - R(u_{\sigma(\tau)\pm}(x, s+T(\tau))) \to 0$ uniformly in (x, s). Further, since R^{-1} is continuous, letting $s=0$ and using Lemma 7, we see that $u_{\sigma(\tau)\pm}(x, T(\tau)) \to v(x)$ uniformly in $x \in [0,1]$ as $\tau \to \infty$. This and the inequality

$$u_{\sigma(\tau)-}(x, \tau - \sigma(\tau)) \leq u(x, \tau) \leq u_{\sigma(\tau)+}(x, \tau - \sigma(\tau))$$

(Lemma 8) prove the desired results. □

5. Final Remarks. The above argument can be applied to the problem with n dimensional spacial variables. In this case it is not easy to verify the Hölder estimates (Lemma 5) of $R(u(x, t))$, which is however inessential to apply the Ascoli-Arzela theorem. Note that in [2] is proved the uniform continuity in $\bar{Q}$ of the function $R(u(x, t))$, from which the equi-continuity of (9) follows.

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