8. Complex Analytic Compactifications of C^3

By Mikio FURUSHIMA
Kumamoto National College of Technology

(Communicated by Kunihiko KODAIRA, M. J. A., Jan. 12, 1988)

Let X be an n-dimensional connected compact complex manifold and Y an analytic subset of X. We call the pair (X, Y) a complex analytic compactification of C^n if $X-Y$ is biholomorphic to C^n. For $n=1$, it is easy to see that $(X, Y)\cong(P^1, \infty)$. For $n=2$, Remmert-Van de Ven [6] proved that $(X, Y)\cong(P^2, P^1)$ if Y is irreducible, where $Y=P^1$ is a line in P^2.

In this note, we will study the case in which $n=3$. Our main result is the following

Theorem. *Let (X, Y) be a complex analytic compactification of C^3. Assume that Y is normal. Then $(X, Y)\cong(P^3, P^2)$, (Q^3, Q^2_0) or (V_5, H_5), where $Q^3\hookrightarrow P^4$ is a smooth quadric hypersurface, Q^2_0 is a quadric cone, V_5 is a Fano 3-fold of degree 5 in P^6 and H_5 is a hyperplane section of V_5 which has a singularity of A_4-type.*

Remark. These pairs (P^3, P^2), (Q^3, Q^2_0), (V_5, H_5) really exist.

Sketch of Proof. Let (X, Y) be as in Theorem. The normality of Y implies the projectivity of X (cf. [1]). Then X is a Fano 3-fold with $b_2(X)=1$. Let $r(0<r\leqq 4)$ be the index of X. In the case of $r\geqq 2$, we have proved in [2] the following results:

(i) $r=4\Rightarrow(X, Y)\cong(P^3, P^2)$

(ii) $r=3\Rightarrow(X, Y)\cong(Q^3, Q^2_0)$

(iii) $r=2\Rightarrow(X, Y)\cong(V_5, H_5)$.

Therefore we have only to show that the case of $r=1$ can not occur.

Assume that such a (X, Y) exists in the case of $r=1$. Then, by the classification of Fano 3-folds due to Iskovskih [3] and the detailed analysis of the singularities of the boundary Y, we find that $(X, Y)\cong(V_{22}, H_{22})$, where $X=V_{22}$ is a Fano 3-fold of degree 22 in P^{13} and $Y=H_{22}$ is a hyperplane section of V_{22} which has a minimally elliptic singularity x of type $A_{3,**,o}+D_{5,*,o}$ in the terminology of Laufer [6].

Next, let us consider the triple projection of V_5 from the singularity x of Y. Then we have the diagram below:

$$\begin{array}{ccc} V'_{22} & \xrightarrow{\rho} & W \\ \sigma\downarrow & & \downarrow\pi \\ V_{22} & \xrightarrow[\varphi]{} & Q \end{array}$$

where

(1) $\sigma: V'_{22}\to V_{22}$ is the blowing up of V_{22} at the point $x\in Y\subset X$.

(2) W is a smooth 3-fold and $\pi: W\to Q$ is a conic bundle over a quadric

surface Q in $\boldsymbol{P}^3$.

(3) $\rho: V'_{22} \dashrightarrow W$ is a birational map which is called the "flip".

(4) $\varphi: V_{22} \dashrightarrow Q$ is the triple projection of V_{22} from the point x.

Since $\pi: W \rightarrow Q$ is a conic bundle, by Mori [5], Q is smooth, hence, $Q \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. This is a contradiction, since $b_2(W)=2$. Therefore such a (X, Y) does not exist. Details will be published elsewhere.

The author would like to thank the Max-Planck-Institut für Mathematik in Bonn, especially Prof. Hirzebruch for his hospitality and would also like to thank Dr. N. Nakayama for the stimulating conversations we had and for his valuable comments.

References

[1] Brenton, L.: Some algebraicity criteria for singular surfaces. Invent. Math., **4**, 129–144 (1977).

[2] Furushima, M.: Singular Del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\boldsymbol{C}^3$. Nagoya Math. J., **104**, 1–28 (1986).

[3] Iskovskih, V. A.: Anticanonical models of three-dimensional algebraic varieties. J. Soviet Math., **13**, 745–814 (1980).

[4] Laufer, H. B.: On minimally elliptic singularities. Amer. J. Math., **99**, 1257–1295 (1977).

[5] Mori, S.: Threefolds whose canonical bundles are not numerically effective. Ann. Math., **116**, 133–176 (1982).

[6] Remmert, R. and Van de Ven, T.: Zwei Sätze über die komplex-projektive Ebene. Niew Arch. Wisk., **8**(3), 147–157 (1960).