33. *Deficient Cubic Spline Interpolation*

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1. Introduction. Meir and Sharma [6] have studied the problem of interpolation of matching a cubic spline at one intermediate point and deficient cubic spline at two intermediate points between the successive mesh points. For further results in this direction reference may be made to Dikshit and Rana [5], Chatterjee and Dikshit [3] and Rana [7]. Following Schoenberg [8] and de Boor [2], the problem of deficient cubic spline interpolation has been studied by Dikshit and Powar [4]. Corresponding to foregoing Hermite interpolation problem we shall study in this paper a Hermite Birkhoff interpolation by deficient cubic spline. Interesting studies exhibiting sharp convergence properties for such spline interpolant when $f\in C^3$ or $f\in C^4$ have also been made. Our result, in particular includes the results proved in [7].

2. Existence and uniqueness. Let $P: 0=x_0<x_1<\cdots<x_n=1$ denote a partition of $[0,1]$ with equidistant mesh points so that $h=x_i-x_{i-1}=1/n$. Let P_3 be the set of all real algebraic polynomials of degree not greater than 3. We define the deficient polynomial spline class $S(3,P)$ as

$$S(3,P)=\{s: s\in C^1[0,1],\ s\in P_3 \text{ for each } [x_{i-1},x_i],\ i=1,2,\cdots,n\}.$$

Throughout g will denote a nondecreasing function on $[0,1]$ such that

$$g(x+h)-g(x)=H=\int_0^h dg, \qquad x\in[0,1-h]. \tag{2.1}$$

Setting

$$h^{r+p}A(r,p)=\int_0^h x^r(h-x)^p dg\,; \qquad r,p=0,1,2,3,$$

we observe that as a consequence of (2.1), we have

$$h^{r+p}A(r,p)=\int_{x_{i-1}}^{x_i}(x-x_{i-1})^r(x_i-x)^p dg, \qquad \text{for } i=1,2,\cdots,n.$$

Writing $\theta_i=(x_i+x_{i-1})/2$ for all i, we propose the following:

Problem 2.1. *Given a function $f\in C^1[0,1]$. Does there exist a unique 1-periodic spline $s\in S(3,P)$ which satisfies the interpolatory conditions:*

$$s'(\theta_i)=f'(\theta_i), \qquad i=1,2,\cdots,n, \tag{2.2}$$

$$\int_{x_{i-1}}^{x_i}(f(x)-s(x))dg=0, \qquad i=1,2,\cdots,n? \tag{2.3}$$

Problem 2.2. *For the function f of Problem* 2.1, *does there exist a*

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unique 1-*periodic spline* $s \in S(3, P)$ *which satisfies the interpolatory conditions*:

(2.4) $$s(\theta_i) = f(\theta_i), \qquad i = 1, 2, \cdots, n,$$

(2.5) $$\int_{x_{i-1}}^{x_i} f' dg = \int_{x_{i-1}}^{x_i} s' dg, \qquad i = 1, 2, \cdots, n?$$

We shall answer the Problem 2.1 in the following:

Theorem 2.1. *Let* $f \in C^1 [0, 1]$ *and* g *be a nondecreasing function such that* (2.1) *holds. Then there exists a unique* 1-*periodic spline* $s \in S(3, P)$ *satisfying* (2.2), (2.3).

Remark 2.1. It is interesting to observe that conditions (2.3) and (2.5) reduce to different interpolating conditions by suitable choice of $g(x)$. Thus, if g is a step function with a single jump of one at one point in each of the mesh intervals then the conditions (2.3) and (2.5) reduce to the interpolation at the points of jump.

Proof of Theorem 2.1 It is clear the $s'(x)$ is quadratic, hence in the interval $[x_{i-1}, x_i]$, we may write

(2.6) $$h^2 s'(x) = h(x - x_{i-1}) m_i + h(x_i - x) m_{i-1} + (x - x_{i-1})(x_i - x) c_i$$

where $m_i = s'(x_i)$ and c_i is an appropriate constant which has to be determined. Using the interpolatory condition (2.2) we notice that

(2.7) $$4 f'(\theta_i) = 2(m_i + m_{i-1}) + c_i$$

This determines c_i and hence by integration, we obtain from (2.6),

(2.8) $$\begin{aligned} 6h^2 s(x) = {} & 3h(x - x_{i-1})^2 m_i - 3h(x_i - x)^2 m_{i-1} \\ & - c_i(2x + x_i - 3x_{i-1})(x_i - x)^2 + 6h^2 d_i \end{aligned}$$

Now using the interpolatory condition (2.3) in (2.8), we have

(2.9) $$\begin{aligned} 3hA(2, 0) m_i - 3hA(0, 2) m_{i-1} - c_i h(3A(1, 2) + A(0, 3)) + 6 d_i H &= 6 \int_{x_{i-1}}^{x_i} f dg \\ &= 6 F_i \end{aligned}$$

say. Since s is continuous at x_i, we have

(2.10) $$6 h m_i + h c_{i+1} = 6(d_{i+1} - d_i)$$

Eliminating c_{i+1}, d_i and d_{i+1} between the equations (2.7)–(2.10) we arrive at the following system of equations:

(2.11) $$(1/h)^3 \left[m_{i+1} \int_0^h \alpha_{i+1}(x, h) dg + m_i \int_0^h \alpha_i(x, h) dg + m_{i-1} \int_0^h \alpha_{i-1}(x, h) dg \right] = F_i^*$$

where

$$\alpha_{i+1}(x, h) = x^2(4x - 3h), \alpha_i(x, h) = h(h^2 + 6x(h - x)), \alpha_{i-1}(x, h) = (h - x)^2(h - 4x)$$

and

$$F_1^* = 6(F_{i+1} - F_i)/h + 4(3A(1, 2) + A(0, 3))(f'(\theta_{i+1}) - f'(\theta_i)) - 4H f'(\theta_{i+1}).$$

In order to prove Theorem 2.1, we shall show that the system of equations (2.11) has a unique set of solutions. Now, the excess of the coefficient of m_i over the sum of the absolute values of the coefficients of m_{i-1} and m_{i+1} in (2.11) is greater than or equal to

$$(1/h^3) \int_0^h (\alpha_i(x, h) - |\alpha_{i+1}(x, h)| - |\alpha_{i-1}(x, h)|) dg$$

in which the integrand equals to

$$\begin{cases} 2x(4x^2-9hx+6h^2)=T_1(x), \text{ say} & x\in[0,h/4] \\ 2h^3 & x\in[h/4,3h/4] \\ 2(h-x)(4x^2+hx+h^2)=T_2(x), \text{ say} & x\in[3h/4,h] \end{cases}$$

which turns out to be >0 for $x\in[0,h]$.

Thus, the coefficient matrix A of the system of equations (2.11) is diagonally dominant and hence invertible. This completes the proof of Theorem 2.1.

In order to answer problem 2.2 we first use the interpolatory conditions (2.5) and (2.4) in (2.6) and (2.8) respectively to get

$$A(1,0)m_i+A(0,1)m_{i-1}+A(1,1)c_i=\int_{x_{i-1}}^{x_i} f'dg=F'_i \tag{2.12}$$

say, and

$$24f(\theta_i)=3h(m_i-m_{i-1})-2hc_i+24d_i. \tag{2.13}$$

Now using (2.12)–(2.13) in (2.10), we get

$$\begin{aligned}[3A(1,1)-2A(0,1)]m_{i-1}+[18A(1,1)-2A(1,0)-2A(0,1)]m_i\\ +[3A(1,1)-2A(1,0)]m_{i+1}=2F_i^{**}\end{aligned}$$

where $F_i^{**}=12A(1,1)h^{-1}\{f(\theta_{i+1})-f(\theta_i)\}-(F'_{i+1}+F'_i)$.

Now following closely the foregoing proof of Theorem 2.1 we prove the following:

Theorem 2.2. *Let g be a nondecreasing function such that* (2.1) *holds and moreover* $g(x_{i-1})=g(x_{i-1}+2h/9)$, $g(x_i-h/3)=g(x_i)$, $i=1,2,\cdots,n$. *Then for any* $f\in C^1[0,1]$, *there exists a unique* 1*-periodic spline* $s\in S(3,P)$ *satisfying* (2.4), (2.5).

3. **Error bounds.** In this section, we shall obtain bounds for the function $e=s-f$ where s is the deficient spline interpolant of f in $S(3,P)$. Given any function g we write for convenience $g(x_i)=g_i$ and $w(g,h)$ for the modulus of continuity of g. Let us write the equations (2.11) as

$$AM=F^*. \tag{3.1}$$

where A is the coefficient matrix and M and F^* are single column matrices (m_i) and (F_i^*) respectively. It may be observed that (cf. [1], p. 21) the row-max norm:

$$\|A^{-1}\|\leqslant R(h), \tag{3.2}$$

where $R(h)=\max\{R_1^{-1},1/2H,R_2^{-1}\}$ with $R_i=(1/h^3)\int_0^h T_i(x)dg$ for $i=1,2$.

From (3.1), we obtain the system of equations for e'_i as follows

$$A(e'_i)=(F_i^*)-A(f'_i)=(U_i), \tag{3.3}$$

say. Now considering $f\in C^4[0,1]$ and using the results that $f(x)=f_j+(x-x_j)f'_j+(x-x_j)^2f''_j/2+(x-x_j)^3f'''_j/6+(x-x_j)^4f^{(IV)}(\beta_j)/24$ and $f'(x)=f'_j+(x-x_j)f''_j+(x-x_j)^2f'''_j/2+(x-x_j)^3f^{(IV)}(\alpha_j)/6$, where β_j and α_j lie in appropriate intervals, we obtain the right hand side of (3.3) as

$$\begin{aligned}12U_i/h^3=&3[A(4,0)f^{(4)}(z_{i+1})-A(0,4)f^{(4)}(z_i)]\\ &+[3A(1,2)+A(0,3)](f^{(4)}(\eta_{i+1})+f^{(4)}(\eta_i))-Hf^{(4)}(\eta_{i+1})\\ &+2[3A(0,2)-6A(1,2)-2A(0,3)]f^{(4)}(\delta_i)\\ &-2[3A(2,0)+6A(1,2)+2A(0,3)-2H]f^{(4)}(\delta_{i+1}).\end{aligned}$$

where $y_i\in[x_{i-1},x_i]$ for $y=z,\eta,\delta$. Rearranging the terms suitably we get

$$12U_i/h^3=3A(4,0)[f^{(4)}(z_{i+1})-f^{(4)}(z_i)]+(H-4A(3,0))[f^{(4)}(\delta_{i+1})-f^{(4)}(z_i)]$$
$$+(2H-12A(1,0)+18A(2,0)-8A(3,0))[f^{(4)}(\delta_i)-f^{(4)}(z_i)]$$
$$+(3A(1,2)+A(0,3)-H)[f^{(4)}(\eta_{i+1})-f^{(4)}(\delta_{i+1})]$$
$$+(3A(1,2)+A(0,3))[f^{(4)}(\eta_i)-f^{(4)}(\delta_{i+1})].$$

Thus,

(3.4) $$|U_i|\leqslant(3H/4)h^3w(f^{(4)},h)$$

Now following the standard arguments based on the diagonally dominant property and using (3.2), (3.4) in (3.3), we have

(3.5) $$\|(e_i)'\|\leqslant(3H/4)h^3R(h)w(f^{(4)},h)$$

where $R(h)$ is given by (3.2).

Now combining (2.7) with (2.6), we replace $s'(x)$ by $e'(x)$ and m_i by e'_i in (2.6) and adjust suitably the additional terms by using the result of Taylor's theorem to see that

(3.6) $$\|e'(x)\|\leqslant 2\|(e_i)'\|+(h^2/2)w(f''',h)$$

Now using the estimate (3.5) in (3.6), we get

(3.7) $$\|e'(x)\|\leqslant(h^2/2)[w(f''',h)+3HhR(h)w(f^{(4)},h)]$$

Thus, we have proved the following:

Theorem 3.1. *Suppose $s(x)$ is the deficient cubic spline of Theorem 2.1 interpolating a function $f(x)$ and $f(x)\in C^4[0,1]$. Then for $r=0,1$*

(3.8) $$\|(s-f)^{(r)}(x)\|\leqslant(h^{3-r}/2)[w(f''',h)+3HhR(h)w(f^{(4)},h)].$$

(3.7) proves Theorem 3.1 for $r=1$. The other inequality of Theorem 3.1 follows by the standard reasoning used elsewhere.

Starting with the system of equations (2.14) and following closely the foregoing proof of Theorem 3.1, we can also prove a similar result corresponding to Theorem 2.2 in the following:

Theorem 3.2. *Suppose $s(x)$ is the deficient cubic spline of Theorem 2.2 interpolating a function $f(x)$ and $f(x)\in C^3[0,1]$. Then for $j=0,1$*

(3.9) $$\|(s-f)^{(j)}(x)\|\leqslant h^{3-j}K(h)x(f''',h)$$

where $K(h)$ is some positive function of h.

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